Polypolyhedra in Origami

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3rd International Conference on Origami in
Science, Math, & Education

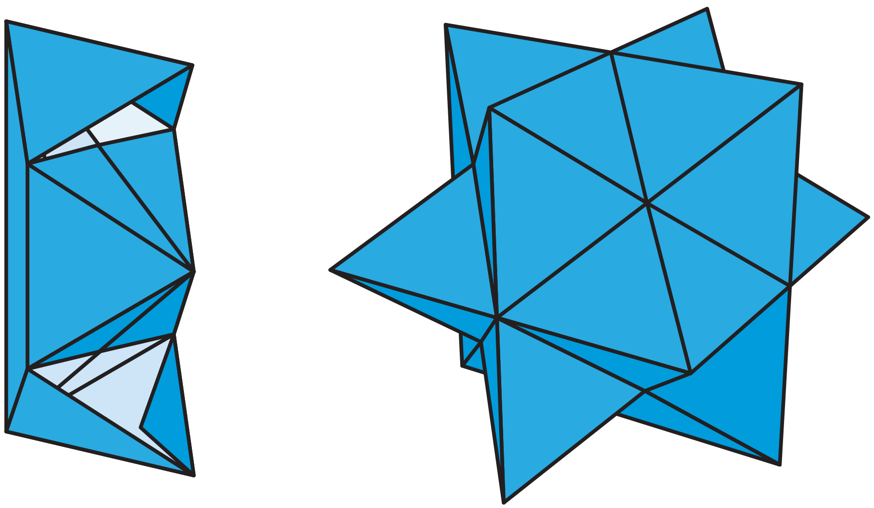
Background

- Origami modulars: many identical pieces that hold themselves together, usually forming a geometric solid
- A “burr puzzle” is a wooden puzzle made of sticks that lock together, usually forming a geometric solid
- Can one make an origami burr puzzle?

- Yes...

Origami Burrs

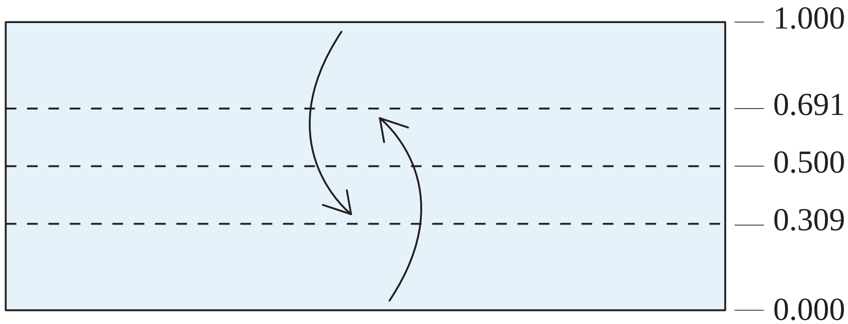
- Peter Ford made several back in the ‘80s
- Here is a rendition of the “classic six-piece burr”



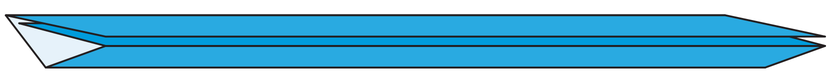
To be published in “Puzzlers Tribute,” David Wolfe, ed. (2001: A. K. Peters)

More Burrs

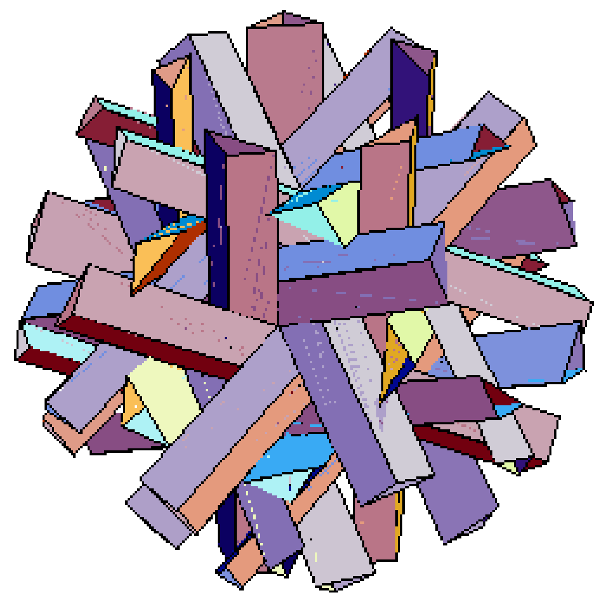
- Can one make a burr with more (but identical) pieces?
- Yes…(but there's one problem)...



1. Begin with a dollar bill. Mark off three horizontal creases at the divisions shown and fold on all three creases so that the ends overlap one another.



2. Finished unit. The stick has a triangular cross section.

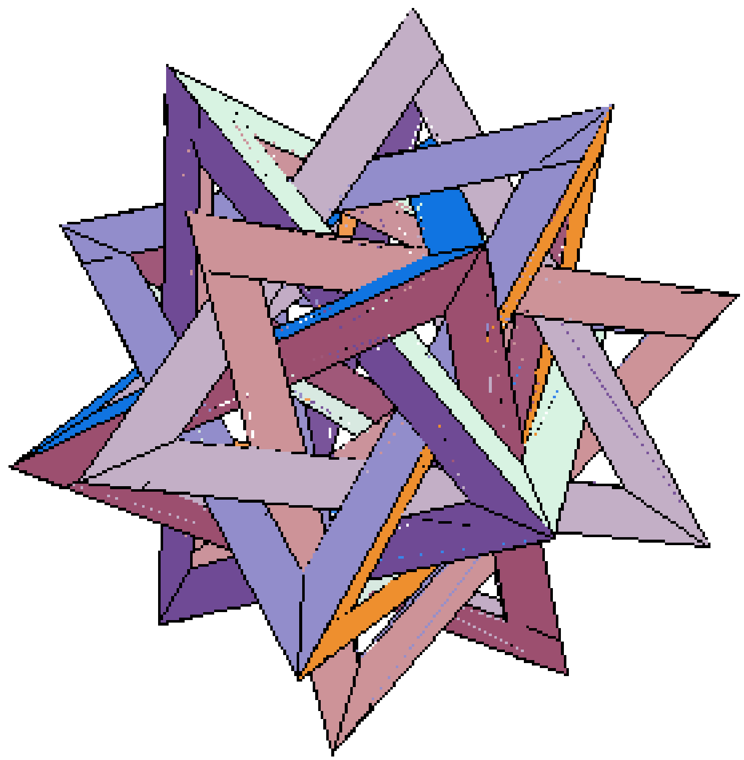


More Burrs

- It collapses!
- No burr composed of convex polyhedra is stable against all possible rigid-body motions.
- What if we joined the sticks at their ends?
- Individual groups of sticks would form polyhedra. The burr would be a collection of interlocking polyhedra.
- Only one problem…

More Burrs

- It's been done!
- Tom Hull's "Five Intersecting Tetrahedra," using Francis Ow's edge unit.



Polypolyhedra

- Are there others?
- What are we looking for?
- A Polypolyhedron is:
  - a compound of multiple linked polyhedral skeletons with uniform nonintersecting edges
  - "Uniform" = all alike
  - The edges are 1-uniform (1 type of edge)
  - The vertices could be 1-uniform or 2-uniform. If 2-uniform, every edge has 1 of each kind of vertex

- How many topologically distinct polypolyhedra are there?

Polypolyhedra

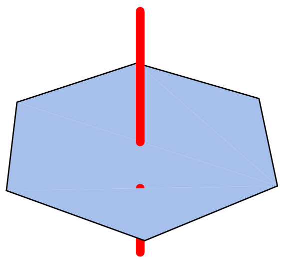
- Prior art
- Five intersecting tetrahedra is a well-known geometrical construction
- “Orderly Tangles” by A. Holden identifies numerous compounds of polygons and polyhedra, which he calls “polylinks”
- No complete listing known

Polypolyhedra & Symmetry

- We start by classifying polypolyhedra according to their symmetry
- If all edges are "alike", then any one edge can be transformed into any other edge by some solid rotation
- Any polypolyhedron can be labeled by the set of rotations that maps any one edge into all others
- The set of rotations for any given polypolyhedron form a Group.

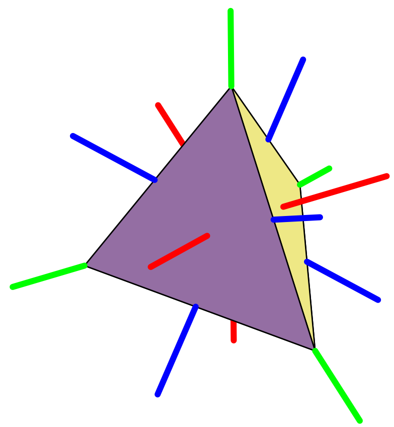
Rotation Groups

- There are 5 families of rotation groups in 3-D



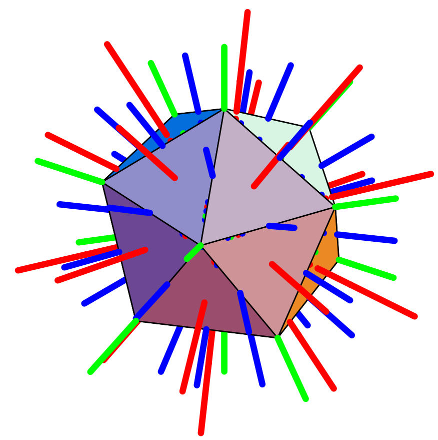
Cyclic group

Order n



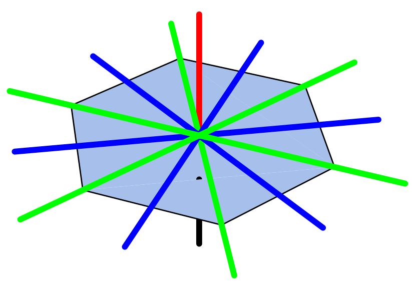
Tetrahedral group

Order 12



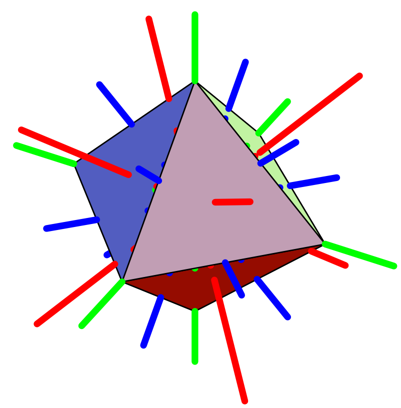
Icosahedral group

Order 60



Dihedral group

Order 2n



Octahedral group

Order 24

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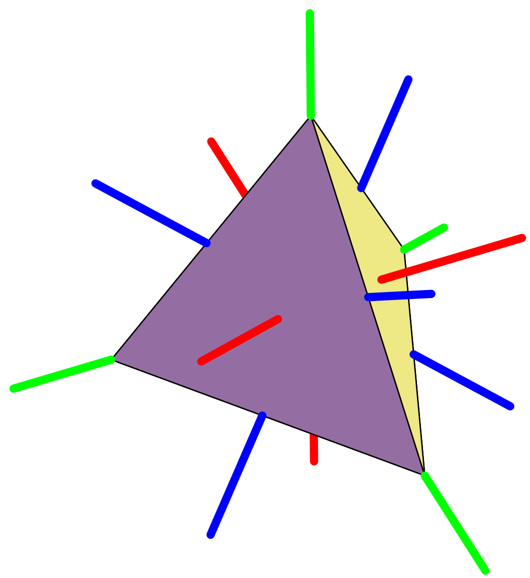
Education

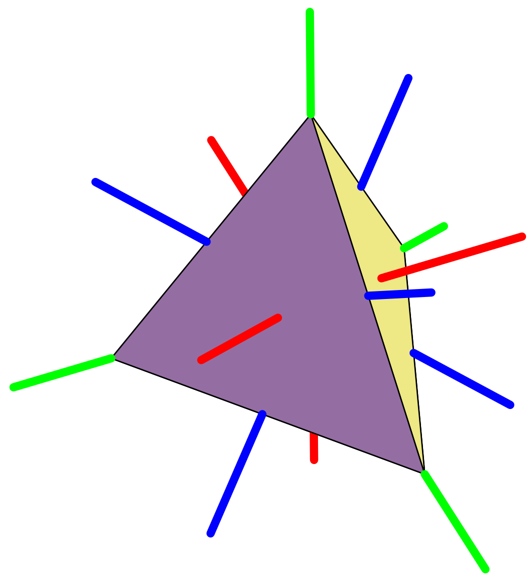
Orbits

- The set of points obtained by applying all rotation operators to a single point is called an “orbit”
- The number of distinct points in an orbit is called the order of the orbit
- The maximum order of an orbit is the order of the rotation group.



Orbits

- If a point is unchanged by a rotation, it lies on an axis of rotation
- The orbits of points on axes of rotation have lower order than the group order
- The tetrahedron has orbits of order 4 (red), 4 (green), 6 (blue), and 12 (everything else).
- Label an orbit by its type: Vertex, Edge, Face, or Complete (V,E,F,C)



Orbits

- Every vertex of every edge of a polypolyhedron must lie in an orbit
- We can classify polypolyhedra by their rotational symmetry and by the orbits of the two vertices of any edge.

	V	E	F	C
Cyclic(n)			1	n
Dihedral(n)	n	n	1	2n
Tetrahedral	4	6	4	12
Octahedral	6	12	8	24
Icosahedral	12	30	20	60

Orbital Types

- Label the edge by a symbol {o1,o2} of the orbital types of its vertices
- For example, {V,C} is an edge with one vertex in a V (vertex) orbit, the other in a C (complete) orbit
- Append a number if the orbits are of the same type but different orbits: {V1,V1} mean the vertices are in two different V orbits

Orbital Types

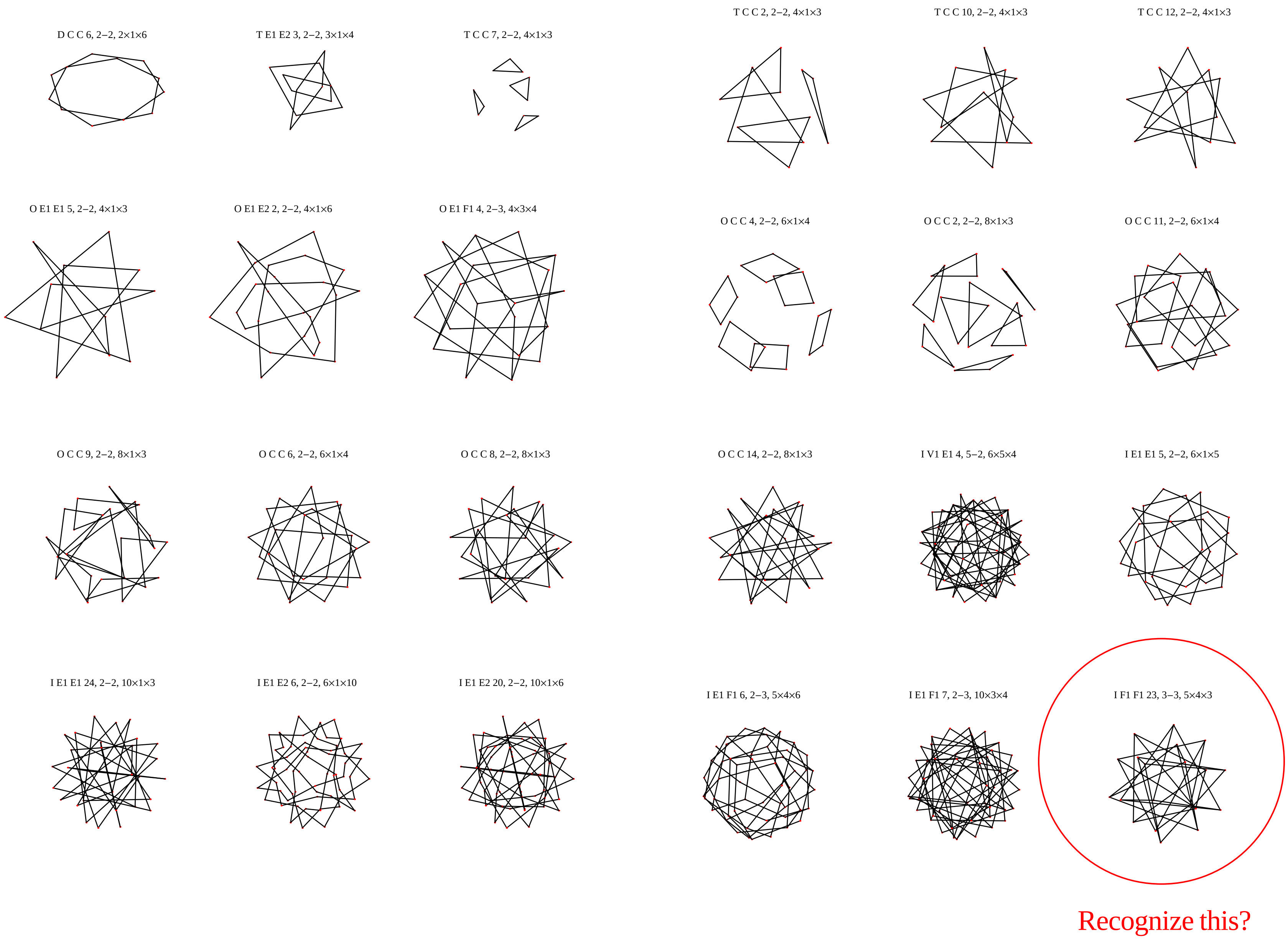
- Enumerate orbits
- Eliminate redundancies, e.g., {V,C} is the same as {C,V}
- Eliminate orbits that lead to 1-valent edges (disjoint sticks), e.g., {V,C}

- There are now only 10 distinct types of edges that lead to interesting polypolyhedra: {V1,V1}, {V1,V2}, {V1,E1}, {V1,F1}, {E1,E1}, {E1,E2}, {E1,F1}, {F1,F1}, {F1,F2}, and {C,C}.

Seed Edges

- For each orbital type {o1,o2}, pick a single vertex in o1.
- Every possible choice of o2 vertex gives a different “seed edge.”
- Apply all rotation operators to get all other edges.
- Eliminate duplicates

- There are 41 distinct families!



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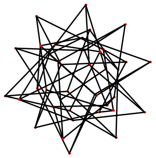
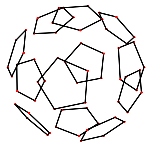
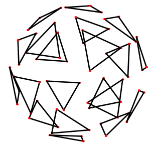
I F1 F2 7, 3−3, 5×6×4



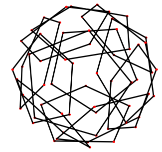
I C C 14, 2−2, 12×1×5



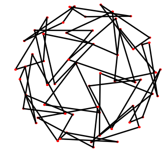
I C C 3, 2−2, 20×1×3



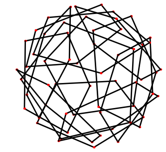
I C C 6, 2−2, 12×1×5



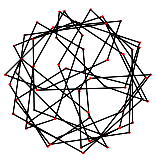
I C C 32, 2−2, 20×1×3



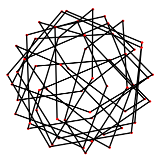
I C C 4, 2−2, 12×1×5



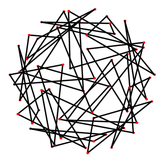
I C C 28, 2−2, 12×1×5



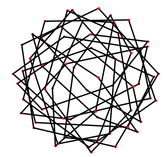
I C C 9, 2−2, 12×1×5



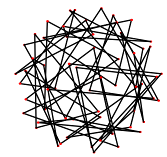
I C C 12, 2−2, 20×1×3



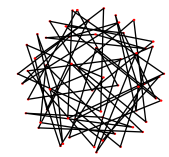
I C C 11, 2−2, 12×1×5



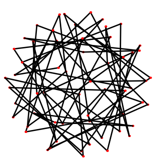
I C C 40, 2−2, 20×1×3



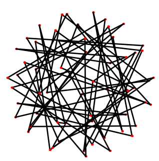
I C C 60, 2−2, 20×1×3



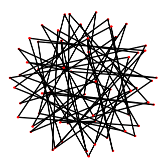
I C C 8, 2−2, 20×1×3



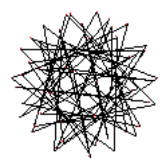
I C C 35, 2−2, 20×1×3



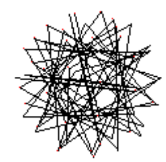
I C C 24, 2−2, 20×1×3



I C C 23, 2−2, 20×1×3



I C C 36, 2−2, 20×1×3



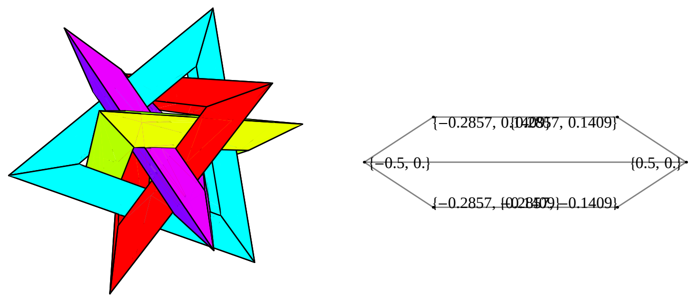
From Skeleton to Origami

- Not all polypolyhedra are linked; eliminate unlinked
- Some belong to families characterized by 1 or 2 continuously-variable parameters
- But, we can enumerate them all!

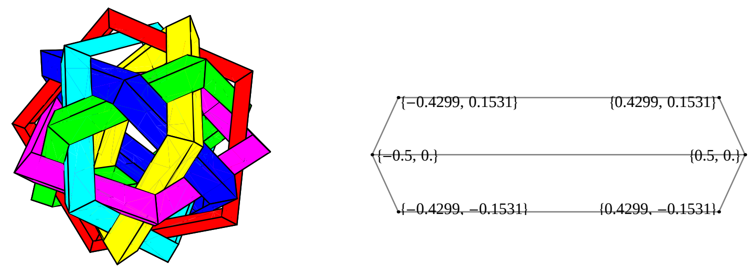
- Replace each line in the skeleton with the widest possible edge unit (e.g., an Ow unit)

Family 1: Discrete

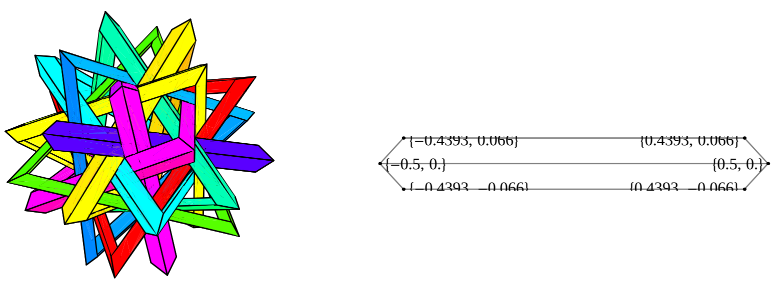
- There are 4 distinct homoorbital polypolyhedra



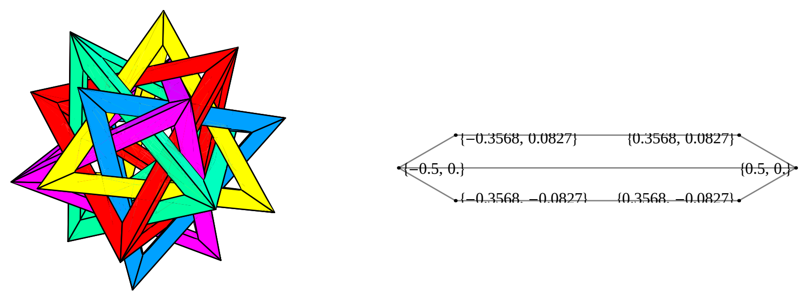
Four Intersecting Triangles (4x1x3)



Six Intersecting Pentagons (6x1x5)



Ten Intersecting Triangles (10x1x3)



FIve Intersecting Tetrahedra (5x4x3)

Family 2: 1-parameter variation

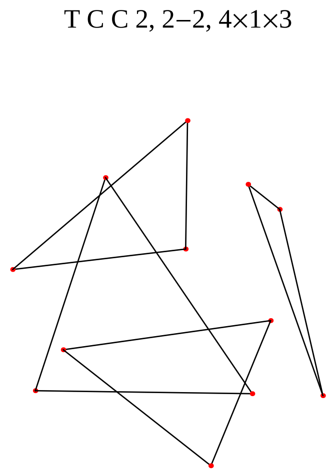
- Heteroorbital polypolyhedra are 2-uniform, 2 types of vertex composing two orbits
- Each forms a continuously varying family characterized by the ratio of their orbital radii
- Different orbital ratios can be topologically distinct ("varieties")
  - T {E1 E2} - (3x1x4) - 1 variety
  - O {E1 E2} - (4x1x6) - 1 variety
  - O {E1 F1} - (4x3x4) - 2 varieties
  - I {V1 E1} - (6x10x4) - 4 varieties
  - I {E1 F1} - (5x4x6) - 2 varieties
  - I {E1 F1} - (10x3x4) - 3 varieties

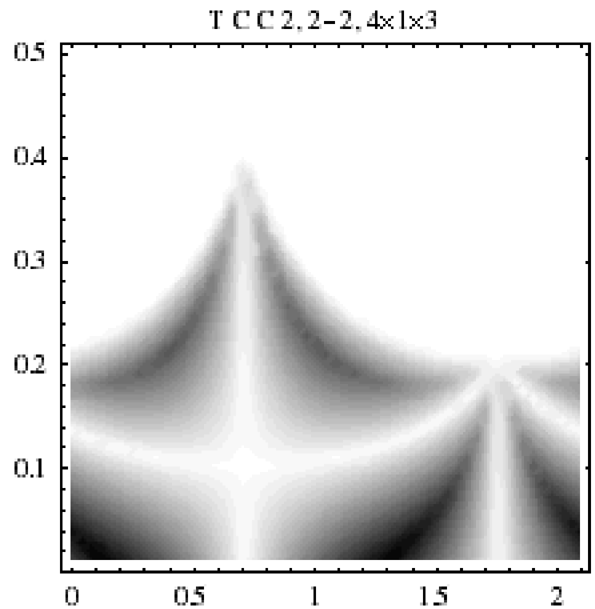
Family 3: 2-parameter variation

- The orbital seed vertex of a {C,C} polypolyhedra can vary in 2 dimensions
- There are 5 families, each corresponding to a manipulation of a Platonic solid
- Shift each face outward by a factor ρ, rotate through an angle ϕ.
- Plot the distance of closest approach between any two edges versus ρ and ϕ.

{C,C} Tetrahedron

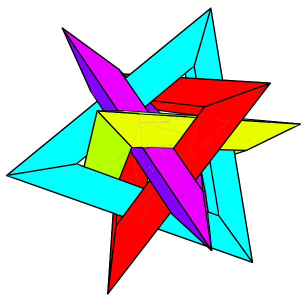
- Shift and rotate each face
- Plot the distance of closest approach between edges
- Each dark region is a topologically distinct polypolyhedron

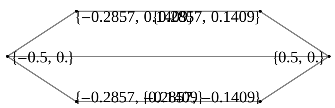


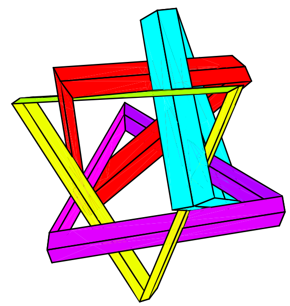


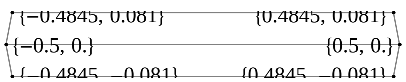
{C,C} Polytetrahedron

- There are 2 topologically distinct {C,C} polytetrahedra.



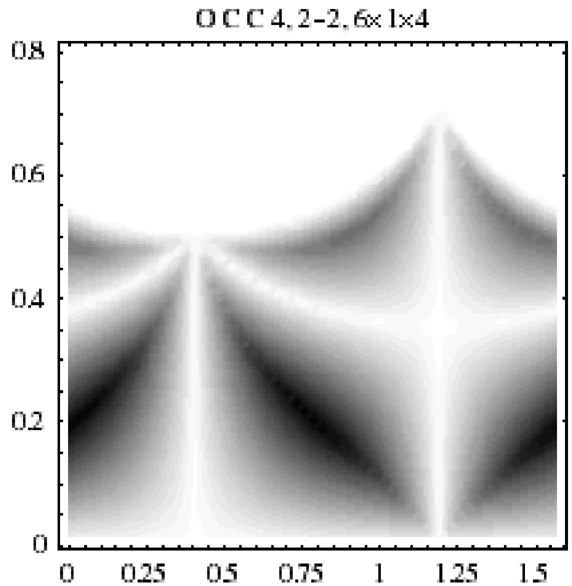


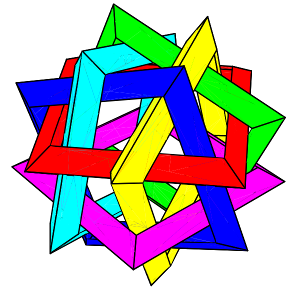


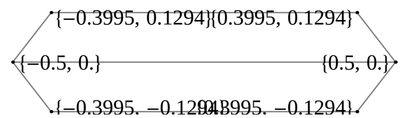


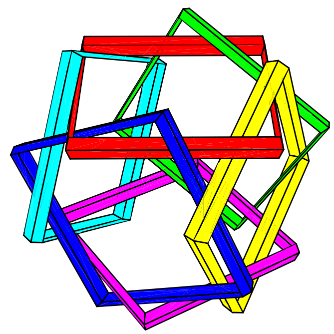
{C,C} Polyhexahedra

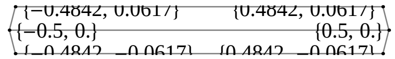
- There are 2 distinct Polyhexahedra (6x1x4).



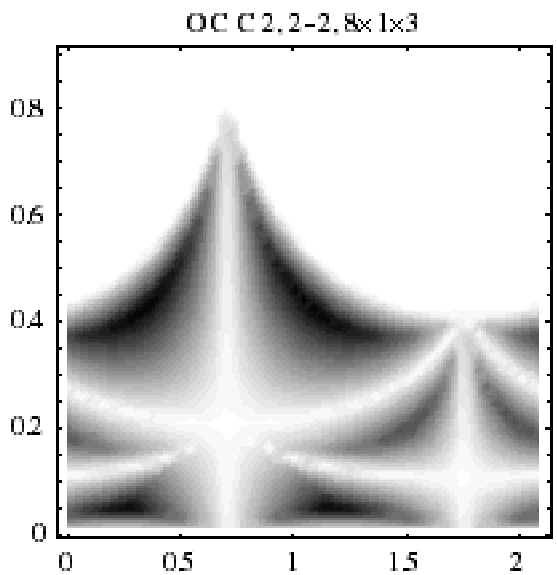


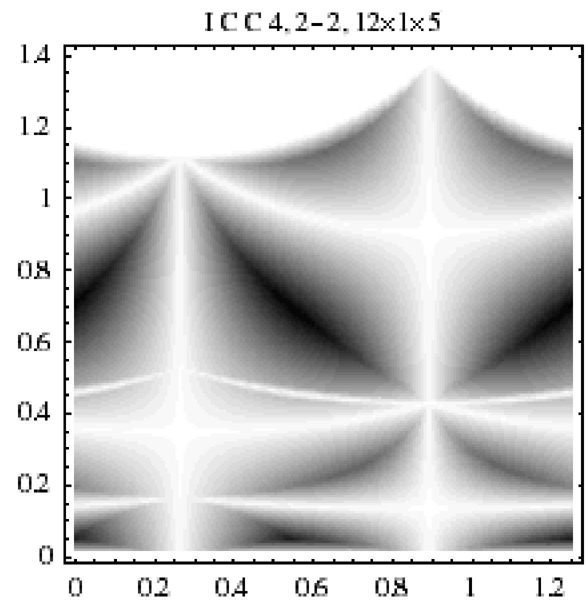






Polyoctahedra, Polydodecahedra



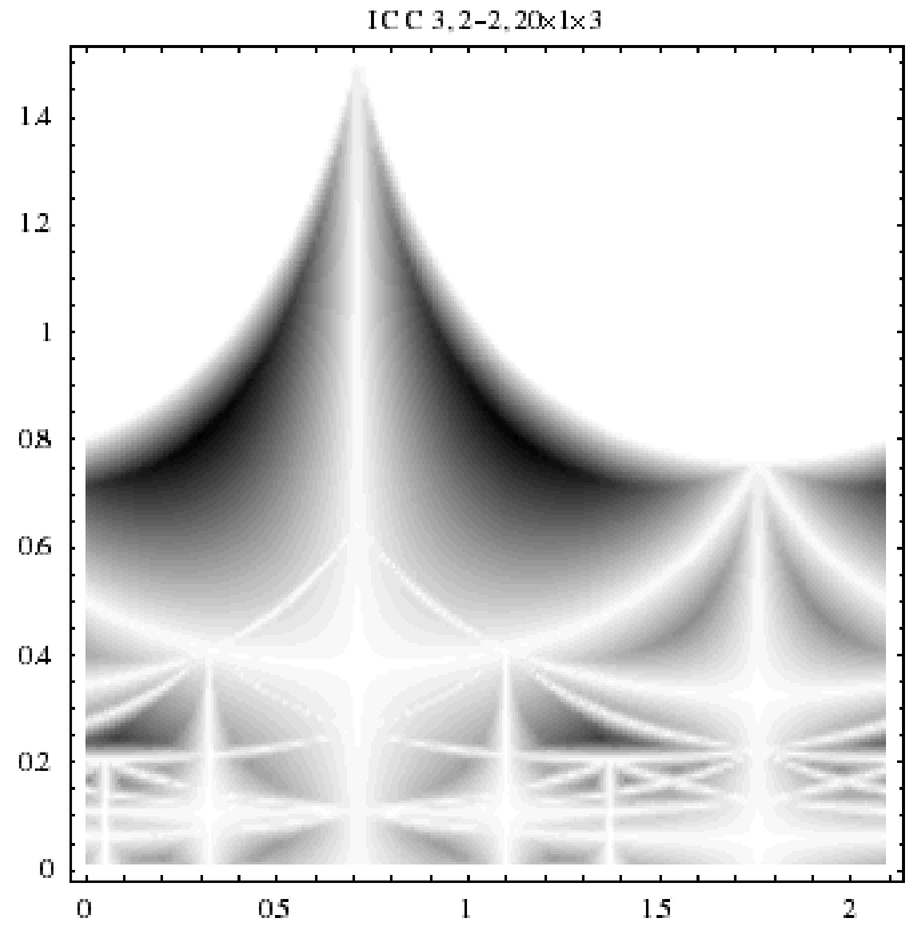


3 varieties of polyoctahedra

5 varieties of polydodecahedra

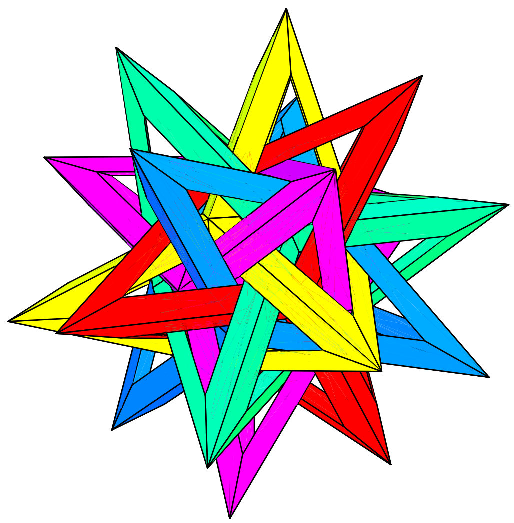
Polyicosahedra

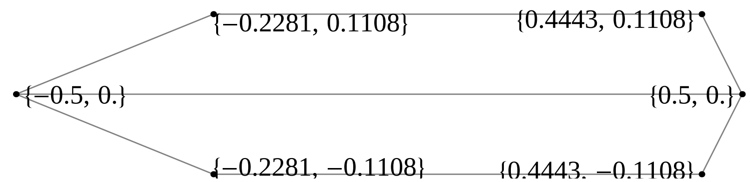
- There are 23 polyicosahedra
- A total of 54 polypolyhedra!
- Nearly all have 2-valent vertices (polygons, rather than polyhedra) with two exceptions:
  - Five intersecting tetrahedra
  - And…

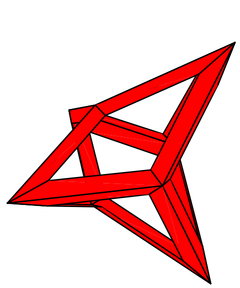
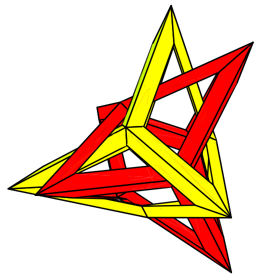
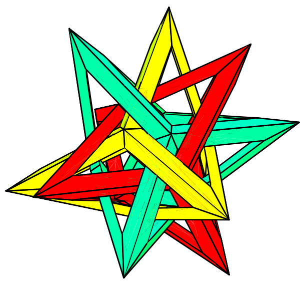
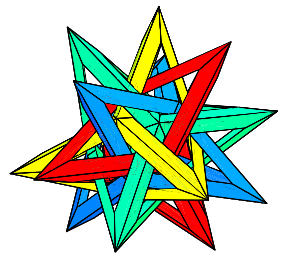


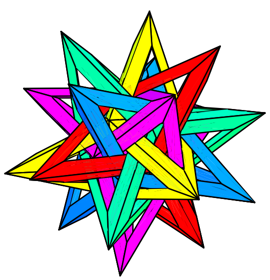
Five Intersecting Rhombic Hexahedra

- I {F1 F2}, 5x6x4, sixty identical edge units







Instructions: OrigamiUSA 2000 Annual Convention Program

First Full Assembly: Peter Budai (photo)