Origami Burrs and Woven Polyhedra

by Robert J. Lang

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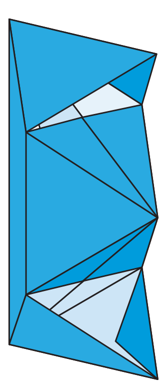
In 1999 I attended the 2nd Scandinavian Origami Convention in Stockholm, where one of the attendees was Herman van Goubergen. Herman had some new models to display. This was an occurence of some note, because Herman only invents about one new model per year, and every time he does, it's utterly unlike anything you've ever seen before in the origami world. This year was a particularly auspicious occasion: he had invented *several* models.

Herman's models were modulars. But, in keeping with his *modus operandi*, they were very different from any other modulars I'd ever seen, making use of curves and novel locking mechanisms. Herman explained his motivation: "I was trying to come up with modulars that were not just another way of making a triangle by sticking flaps into pockets." And that got me to thinking about modulars, too.

I periodically play around with modulars, attracted by their geometrical properties and three-dimensionality. The way it works is, I get intrigued by their structure and design one or two; but then I have to actually fold it, and after I've folded what seems like several thousand units I ask myself why I'm singing my wings against this particular flame and go back to folding something restful and relaxing, like color-changed mating mosquitos.

What Herman got me to thinking was, "what other ways are there to build modulars?" This thought bumped into another question that's been rattling around my brain lately, which is what to do for the *Gathering For Gardner 4* meeting next spring. *Gathering for Garder 4* (G4G4) is an eclectic meeting of mathematicians, magicians, and puzzlists bound by a common interest in the hobbies of Martin Gardner. At G4G3 a few years ago, I met a boatload of interesting characters, wound up at a dinner table full of burr-puzzle-makers, and ended up trading origami books for wooden burr puzzles, one of which I reproduced in origami and published in the OUSA 1998 Annual Collection.



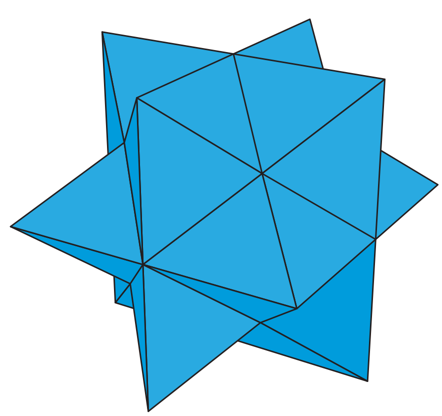


Figure 1. Left: a single piece from an origami burr. Right: the assembled origami burr puzzle.

"Aha!" I thought. "What other burr puzzles can be done with origami?" Now burr puzzles in origami are not actually new — Peter Ford designed and diagrammed several many years ago. But then my standards for novelty aren't nearly as high as Herman's. In fact, it is not uncommon that I see that someone has found a nice golden model, and even though he or she may have picked up the obvious nugget, I have no compunctions about bringing in the dredge and sluice box to see if there are any other interesting bits left behind. Sometimes, all I get are mounds of origami tailings; but occasionally, there are other nice nuggets -- and if I get lucky, great big fat veins -- of origami designs to be found.

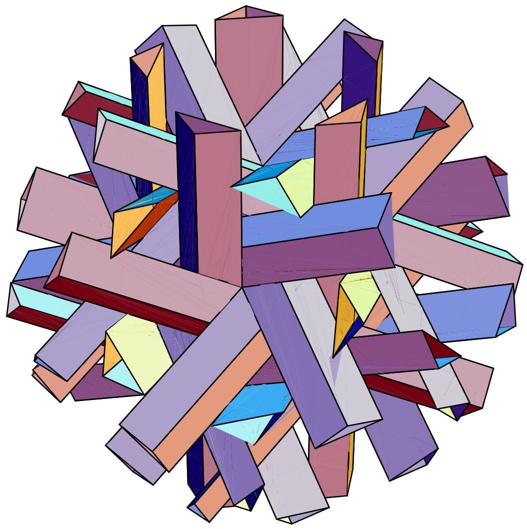
So, I thought about origami burr puzzles. In case you're not familiar with them, burr puzzles are puzzles made up of sticks that fit together to make interesting star-shaped solids. (They're called "burr" puzzles because all the sticking-out bits make them resemble the burrs one finds in one's socks after an outing in the wild.) The sticks generally have the cross section of some simple polygon and they usually have notches cut into them that determine how they fit together. Although the pattern of notches can vary from stick to stick within a single burr, the most interesting puzzles (to me, at least) have all identical sticks. This is particularly the case in a burr destined for an origami implementation: if all the sticks are identical, then I only have to design a single shape for the entire burr puzzle.

Actually, burr puzzles offer another nice property as an origami subject: a complicated-looking model using only simple folds. Origami modulars are one of the few stylistically acceptable ways to use multiple sheets of paper. The individual piece parts can be very simple, but if you use a lot of them, the finished model still looks challenging and complex.

So I figured I could make some pretty neat origami models by folding and assembling many copies of a shape that mimicked a piece of a burr puzzle -- a (possibly) notched polygonal stick. But then my next thought was that while making a stick with a polygonal cross section would be a cinch, putting one or more notches into the stick caused a couple of problems. First, putting a notch would require putting some pleats into the stick (to get the excess paper to form the notch). Those pleats would break up the clean lines of the stick and make it harder to fold. Second, the most obvious way of adding notches was already used by Peter Ford in his burr origami, and I was hoping for at least a *little* novelty. So I thought I'd start out by looking for burr puzzles made up of identical *un-notched* sticks, that is, arrangements of polygonal-cross-section sticks that touch, but don't intersect, one another.

But first, I needed a burr puzzle to start with. Burr puzzles are fundamentally geometrical shapes. When it comes to analysis of things geometrical, the tool of choice for me is the computer program *Mathematica*, the Swiss army knife of scientists and engineers all over the world. It was a fairly straightforward problem to set up: what are some interesting arrangements of polygonal-cross-section sticks that don't intersect? I also set the condition that each stick must be touched by at least 6 other sticks, so that every stick would be held in place firmly by its neighbors.

After a few evenings of playing around, I found the following interesting arrangement of 30 triangular sticks:



Each stick has a triangular cross section and each touches 6 others, which guarantees that if its neighbors are fixed in space, no stick can move in any direction (other than sliding along its length, of course; I was relying on friction to take care of that direction of movement). What was really appealing about this was that the unit was ridulously easy, taking only 3 creases, and could be made from a dollar bill.

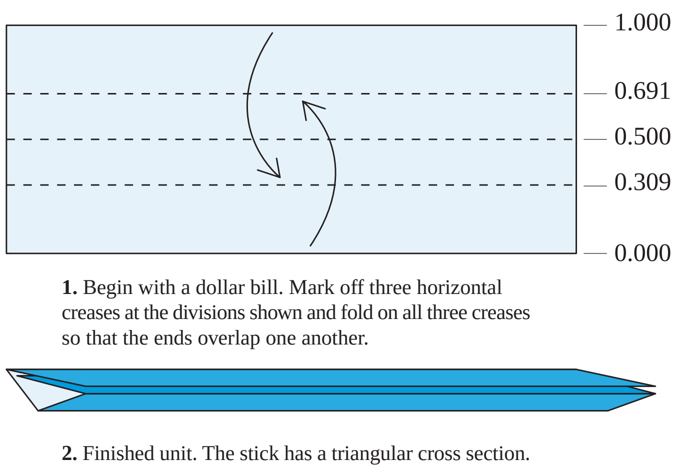


Figure 2. Folding sequence for the unit element of the 30-unit arrangement of sticks.

With pounding heart and thirty bucks, I folded up 30 sticks, carefully assembled them into the figure above, holding them together with bits of tape as I assembled the structure. Then, after the model was assembled, all sticks in place, each wedged tightly against its neighbors, I carefully removed the tape. The model promptly collapsed in a heap. Oops.

What went wrong? I had thought that my design would prevent collapse. Having studied basic Mechanical Design 101, I was certainly aware of the property of kinematic mounting, a theory that says that if you have six point contacts on a shape not all in the same hemisphere it is not free to move in any direction. The same holds true if you have 5 point contacts on a shape with translational symmetry; it can't move in any direction other than the direction of symmetry. In my origami model, each stick was held at six points by its neighbors, so other than sliding (which wasn't the problem) there was no way it could move.

Unless they all moved at the same time. If you look carefully at the rendering of the 30-stick construction, you can see that each of the sticks is held by its neighbors. But if you move each of them microscopically away from the center of the shape, then each no longer touches any other but it's still a valid (nonintersecting) arrangement of sticks. Now they can rattle around a bit and they're all free to move even further away, where they can rattle around some more, and so on, and eventually they collapse in a heap. So the configuration is inherently unstable. In fact, after a bit more reflection, I realized that any such arrangement of identical unnotched sticks is similarly unstable. (This was probably taught in Mechanical Design 102, but I didn't get that far.) At any rate, since the origami world tends to look askance at models that are accompanied by the instruction, "Please don't breathe on this," I determined that this approach was, origamically, a dead end.

But maybe it could be patched up. The structural instability arises if every stick is free to move away from the center. The way to prevent the instability would be to prevent the ability of a stick to move away from the center. Two solutions suggested themselves: (1) wrap lots of rubber bands around the model; (2) join the ends of each stick to something that prevents its motion -- like, say, the end of another stick. The second seemed more aesthetically pleasing.

So the idea would be to make an arrangement where each stick is joined to other sticks at both ends. If all sticks are joined to others, then they must form one or more polyhedral skeletons. Hmmm, I thought, a polyhedral shape made up of interwoven sticks joined to other sticks at their ends. That sounded like a great avenue to pursue. It sounded like a beautiful, interesting structure. It sounded like a tree ripe for plucking! It sounded like, like,..um, er, it sounded like Tom Hull's Five Intersecting Tetrahedra.

Yes, unfortunately for my hopes of novelty, Hull's Five Intersecting Tetrahedra, aka FIT, is a perfect embodiment of the concept. In this model, five skeletal tetrahedra penetrate each other to form a 20-pointed polyhedral star. It is stable. It is very cool. It is also a bit of a challenge to put together, which only increases its cool factor. And it's made from a nice, simple edge unit (one originally developed by Francis Ow, as Tom carefully credits). The polyhedron from which Tom's model is derived is one of the 59 stellations of the icosahedron. The polyhedron itself is quite old, and Tom's version is not the first origami implementation of this particular polyhedron. But it is by far the coolest.

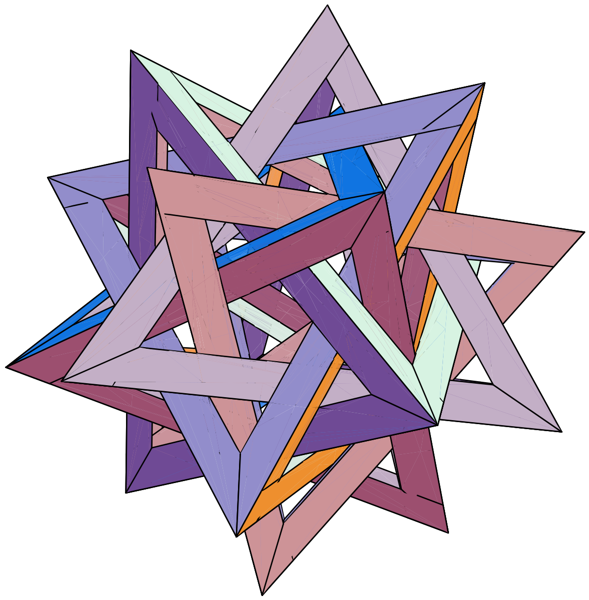


Figure 3. A *Mathematica* rendering of Five Intersecting Tetrahedra.

You can find photographs, instructions, and the mathematics behind Hull's FIT at his web site, http://chasm.merrimack.edu/~thull/gallery/modgallery.html.

Well this was frustrating. Here I had dug my way to a buried treasure, only to find that the tomb had already been plundered by another party. Phooey. But then, the dredge-and-sluice gene in me started kicking in again: there had to be more polyhedra than this, and just maybe, one or more of the others could be rendered with interwoven edges in a way that was as cool (or dare I hope: even cooler?) than Tom's rendition of FIT. But any new polyhedron wasn't going to yield itself to casual inspection. So, I set myself a goal: carry out a systematic analysis of all polyhedra composed of interwoven polyhedra that can be made using a single edge unit.

But how to go about this? Where to even begin? I began by trying to write down what I could about my goal without unnecessarily limiting my search. I wanted to find structures composed of a single edge unit. For stability, each edge unit needs to be pinned by other units touching it along its length and it should be joined to other units at its ends. If every edge unit is identical, then that means that each edge and its immediate environment — the edges around it and touching it — needs to be in the same arrangement for every edge in the structure.

That meant that polyhedron I would be building must have certain symmetry properties. Specifically, its edges must be *uniform*, which is a special way of specifying that they are all alike. To be uniform, all edges must be the same length, but in addition, their surroundings must be identical too. Another way to think of this property is, if you were shrunken down to ant size and were sitting on an edge and looked around you, the view would be the same from every single edge in the structure: you couldn't tell which edge you were sitting on without somehow marking the edges.

Polyhedra whose edges are uniform are called uniform-edge polyhedra. They are related to a well-known set of polyhedra, the which are called simply *uniform* polyhedra. But in this case, the term uniform actually applies to the vertices, not the edges. The uniform polyhedra have all edges the same length, though not necessarily with the same surroundings, and all their *vertices* are identical.

The most famous members of the uniform polyhedra are the Platonic solids, which are called *regular* polyhedra. Regular polyhedra have only one kind of face; their faces are uniform, their edges are uniform, and their vertices are uniform.

There are only five.

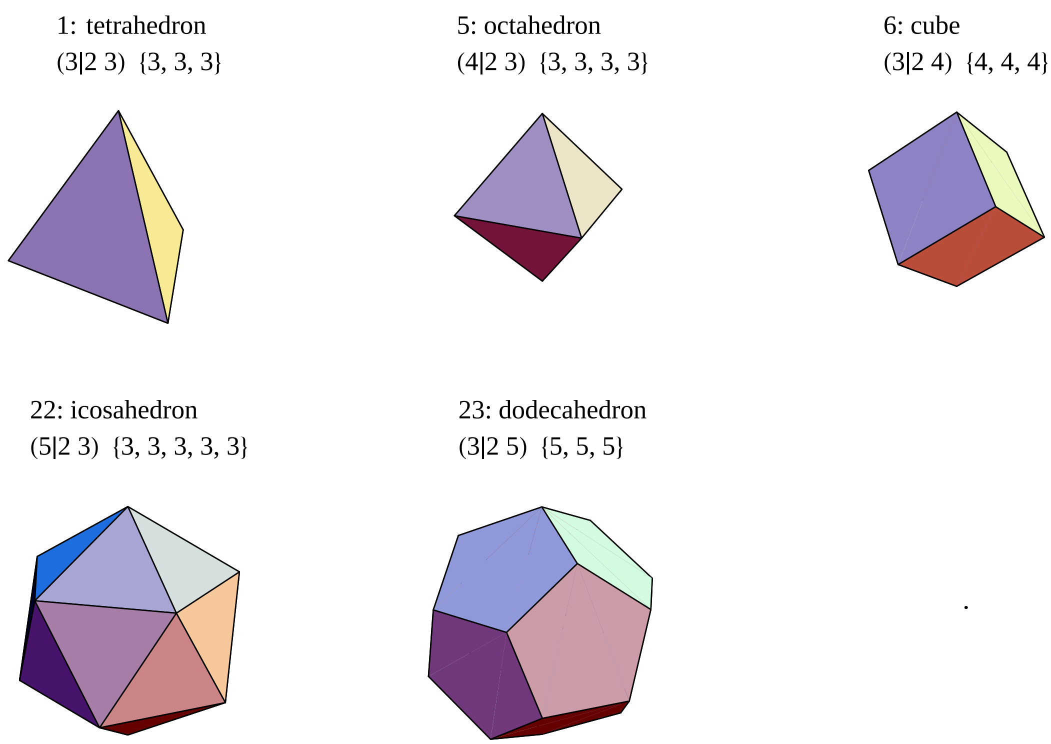
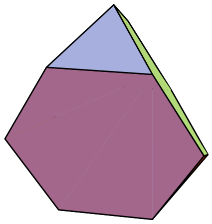


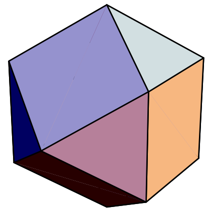
Figure 4. The five Platonic solids.

The uniform polyhedra also include the *semiregular* polyhedra, also called the Archimedean solids, of which there are 13 discrete polyhedra and two infinite families, the polygonal prisms and antiprisms. The polygonal prisms consist of two polygons joined by a ring of squares and there is one for every polygon. The antiprisms consist of two polygons joined by a ring of triangles. In the semiregular polyhedra, the vertices are uniform but the faces aren't uniform and the edges may or may not be uniform. The figure below shows the 13 Archimedean solids, plus one each representative of the prism and antiprism families.

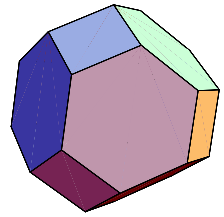
2: truncated tetrahedron
(2 3|3) {6, 6, 3}



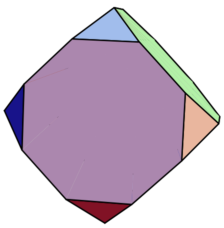
7: cuboctahedron
(2|3 4) {3, 4, 3, 4}



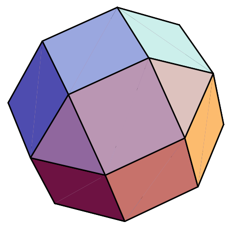
8: truncated octahedron
(2 4|3) {6, 6, 4}



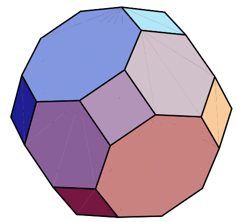
9: truncated cube
(2 3|4) {8, 8, 3}



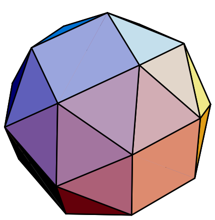
10: rhombicuboctahedron
(3 4|2) {4, 3, 4, 4}



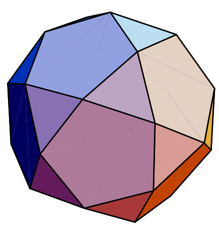
11: truncated cuboctahedron
(2 3 4|) {4, 6, 8}



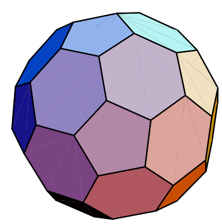
12: snub cube
(|2 3 4) {3, 3, 3, 3, 4}



24: icosidodecahedron
(2|3 5) {3, 5, 3, 5}



25: truncated icosahedron
(2 5|3) {6, 6, 5}



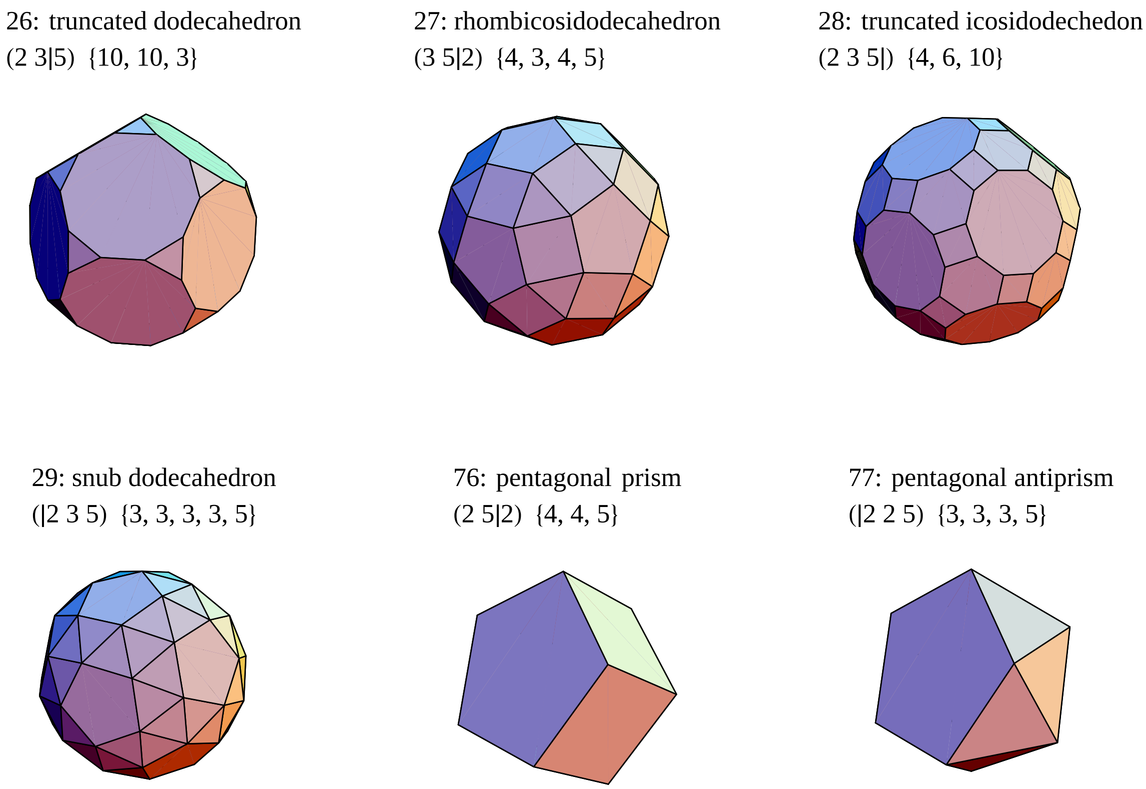


Figure 5. The Archimedean solids, pentagonal prisms and and antiprisms. Both the Platonic and Archimedean solids shown here were computed using Roman Maeder's "UniformPolyhedra.m" *Mathematica* notebook.

The Platonic solids are regular: all their vertices, edges, and faces are uniform. The Archimedean solids have edges all the same length and their vertices are uniform, but their faces are not uniform because there is more than one kind. I was not concerned with whether a polyhedron was regular, semiregular, or uniform, though. I wanted the ones that were uniform-edge (only one type of edge). All of the Platonic solids are uniform-edge, but only two of the Archimedean solids are uniform-edge: the cuboctahedron and the icosidodecahedron. Remember that to be uniform-edge, not only must every edge be the same length, but it must be surrounded by the same types of faces and vertices. In the cuboctahedron, every edge is bordered by a square and a triangle; in the icosidodecahedron, every edge is bordered by a pentagon and a triangle, so both are uniform-edge. In the other Archimedean solids, by contrast, there are two or more types of edge. For example, in the truncated dodecahedron, there are two types of edges: one type lies between two hexagons while the other type lies between a pentagon and a hexagon. Even though the edges all have the same length, the "view" sitting on a hex-hex edge is different from the view from a hex-pent edge; so the polyhedron is not uniform-edge, and therefore it's not a candidate for construction from a single edge unit.

There are also two solids that are uniform-edge but not uniform-vertex; that is, all their edges are alike but there are two types of vertex. These two solids are the rhombic dodecahedron and the rhombic triacontahedron, which have 12 and 30 rhombic faces, respectively. They are the *duals* of the cuboctahedron and the icosidodecahedron. You get the dual of a polyhedron by creating new vertices at the centers of all the faces and then connecting the vertices in adjacent faces with new edges (with some slight distortions to make the faces planar). The rhombic dodecahedron is the dual of the cuboctahedron; the rhombic triacontahedron is the dual of the icosidodecahedron. If you construct the dual of the dual, you find yourself with the original polyhedron you started with. The two rhombic polyhedra, along with rhombic versions of the cube, are the only convex uniform-edge polyhedra with rhombic faces, and, as a point of modular origami trivia, all three can be made from business cards.

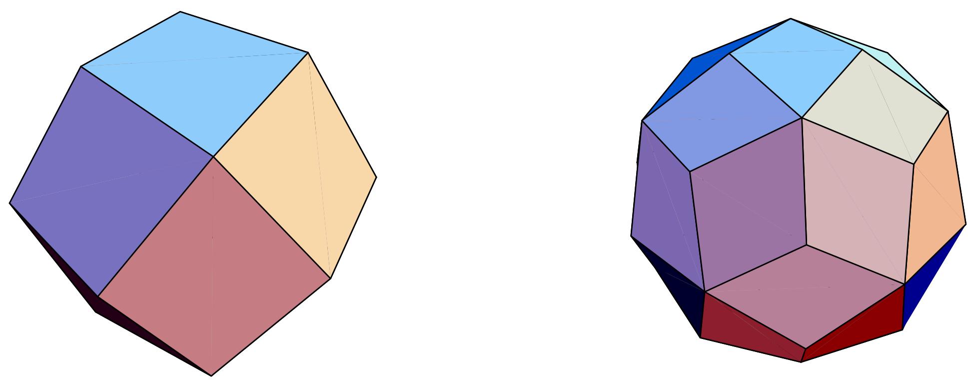


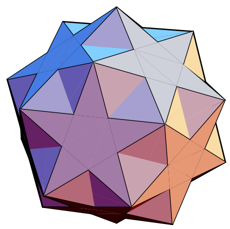
Figure 6. Rhombic dodecahedron and rhombic triacontahedron.

The dual of an uniform-edge polyhedron is also uniform-edge. The Platonic solids have duals, but they turn out to be other Platonic solids. The cube is the dual of the octahedron and vice-versa; the dodecahedron is the dual of the icosahedron; the tetrahedron, uniquely, is its own dual.

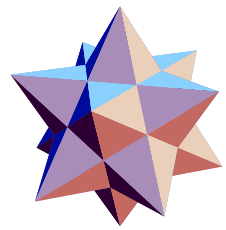
The five Platonic solids plus the two uniform-edge Archimedeans and their uniform-edge duals are all uniform-edge. So I could conceivably make origami polyhedra out of any of them using a single edge unit. But I was after rarer game than ordinary polyhedra; I was after *woven* polyhedra, polyhedra whose faces interpenetrate and whose edges pass by one another. The uniform-edge polyhedra above don't fit this bill. They are all convex; none of the faces penetrate one another; so I was not any closer, it would appear, to achieving my goal.

There's another set of uniform-edge polyhedra, however: which are taken from the non-convex semiregular polyhedra. The non-convex polyhedra are less familiar to most people. They are polyhedra whose faces include non-convex regular polygons, e.g., stars. The pentagram, or 5-pointed star, is such a non-convex polygon. If you draw a 5-pointed star without lifting the pencil from the paper or retracing any lines, you will have drawn a pentagram. It has 5 edges, all the same length. The edges of non-convex polygons intersect one another, and the faces of a non-convex semiregular polyhedron intersect one another as well, which gave me some hope that among them could be found interesting candidate polyhedra. The 5 Platonic solids, 13 Archimedean solids, and 57 non-convex semiregular solids comprise the 75 uniform polyhedra. Of these, 16 are uniform-edge: surely, among those 16 (or their uniform-edge duals), there could be found a polyhedron made from woven edge units? I've already shown the 7 convex uniform-edges; the other 9 are shown below.

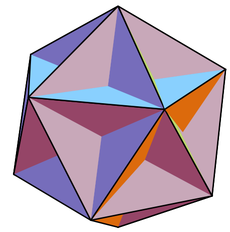
30: small ditrigonal icosidodecahedron
(3|5/2 3) {5/2, 3, 5/2, 3, 5/2, 3}



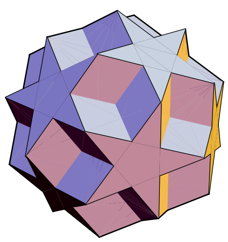
34: small stellated dodecahedron
(5|2 5/2) {5/2, 5/2, 5/2, 5/2, 5/2}



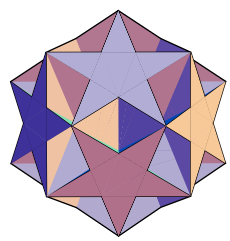
35: great dodecahedron
(5/2|2 5) {5, 5, 5, 5, 5}



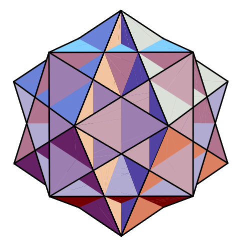
36: dodecadodecahedron
(2|5/2 5) {5/2, 5, 5/2, 5}



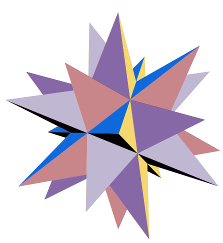
41: ditrigonal dodecadodecahedron
(3|5/3 5) {5/3, 5, 5/3, 5, 5/3, 5}



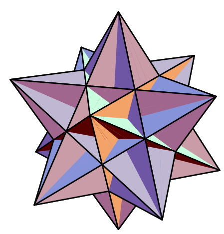
47: great ditrigonal icosidodecahedron
(3/2|3 5) {3, 5, 3, 5, 3, 5}



52: great stellated dodecahedron
(3|2 5/2) {5/2, 5/2, 5/2}



53: great icosahedron
(5/2|2 3) {3, 3, 3, 3, 3}



54: great icosidodecahedron
(2|5/2 3) {5/2, 3, 5/2, 3}

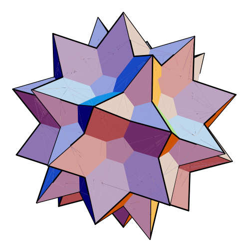


Figure 7. The 9 nonconvex uniform-edge regular and semiregular polyhedra.

It seemed that surely there must be woven polyhedra among the non-convex uniform-edge polyhedra. A quick examination of the polyhedra in figure 7 reveals that what seemed surely the case was not to be. The convex solids were obvious non-starters as interpenetrating polyhedra. But even the non-convex polyhedra, those with intersecting faces, exhibited a fatal flaw: their edges intersect one another as well! But my plan was to build polyhedron from edge units that did not intersect. So this, too, turned out to be a dry hole.

And there was another problem. By this point, I'd turned up a lot of different polyhedra. But the Five Intersecting Tetrahedra are nowhere to be found in this assembly. Not only couldn't my scheme come up with any new intersecting polyhedra — it couldn't even come up with the old ones!

And of course, the reason it wasn't there is that each of the semiregular polyhedra is, by definition, a single connected polyhedron, meaning that by traveling along the edges, you can get from any vertex to any other vertex. Whereas the five intersecting tetrahedra are truly 5 separate polyhedra. If you're on one tetrahedron, you can't get to any of the others by traveling along edges. So there was no way that FIT could be found by examining simple polyhedra. It's just not a simple polyhedron — so it couldn't be anywhere in my list.

Or could it?

There was a tantalizing connection. It might not have been in the list directly, but FIT has a lot in common with some of the polyhedra on the list. As Tom has pointed out, the vertices of FIT are the same as the vertices of a dodecahedron. Both FIT and a dodecahedron have 30 edges. And perhaps, more fundamentally, certainly more importantly, they share the same rotational symmetries. If a given rotation leaves a dodecahedron unchanged, the same rotation leaves FIT unchanged. This similarity holds for every possible rotation of the dodecahedron.

At this point, I started to think of FIT a little bit differently. FIT was not really five separate tetrahedra. FIT was really a dodecahedron — in disguise. Something had been done to this dodecahedron to change it, to make it look like a group of different polyhedra; some transformation that turned the dull, ordinary, everyday dodecahedron into the marvelous, interwoven, FIT. That transformation could be — must be — the key to unlocking the secret of intersecting polyhedra.

It turns out to be a pretty simple transformation. Imagine a line drawn from the center of the dodecahedron through the midpoint of each edge. Think of each line as a axis about which the edge can rotate (like a propeller). We then rotate each each edge counterclockwise through the same angle about its axis. At the same time, we lengthen the edge. Such a transformation is characterized by two numbers: the rotation angle and the lengthening factor. You will find that if you rotate each edge by 45° and lengthen it by a factor of $\sqrt{2}$, the edges meet up with edges other than their former neighbors, forming five new and distinct polyhedra. The polyhedra are tetrahedra; the resulting shape is FIT. You can see this in figure figure below; each red edge is a rotated and lengthened version of the dodecahedron edge that shares its midpoint.

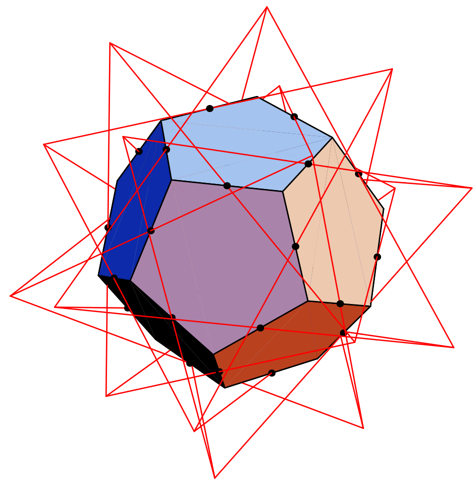


Figure 8. A dodecahedron and the Five Intersecting Tetrahedra.

But this surely isn't the only angle/magnification combination that gives intersecting polyhedra. How could I find others? Here is one way. Picture, for a moment, the dodecahedron where each edge is made arbitrarily long. We slowly rotate each edge at the same rate. We pause every time an edge hits another edge; then we "snip off" the part of the edge that extends beyond the point of intersection. At that point, we'd have an uniform-edge (all edges equal) polyhedron, where the edges are connected at their ends but don't touch any other edges in between the ends.

This basic prescription could be applied to each of the uniform-edge polyhedra to turn them into groups of intersecting polyhedra. But it doesn't necessarily need to be applied to all of them. For example, if you start with a dodecahedron and rotate all the edges by 90°, you wind up with an icosahedron. As you spin the edge from 0° to 90° and back to 180°, you will cycle from dodecahedron through icosahedron and back. So if you apply this procedure to any uniform-edge polyhedron and examine all possible angles, you will have in effect also applied the procedure to the polyhedron's dual. With that in mind, I decided to focus my attentions on the simplest distinct uniform-edge polyhedra: the tetrahedron, octahedron, cuboctahedron, dodecahedron, and icosidodecahedron, which have 6, 12, 24, 30, and 60 edges, respectively.

The next step was a practical problem: how could I tell which rotation angles got all the edges to connect up with others? This conundrum was somewhat simplified by the realization that since the polyhedron was uniform-edge, I only needed to pay attention to a single edge, secure in the knowledge that whatever happened to this edge applied equally to every other edge. So if I rotated all of the edges equally and found a rotation angle where one edge met up with two others, I could be assured that every other edge was also meeting up with two others at exactly the same angle and exactly the same place along each edge.

With this in mind,I used *Mathematica* to set up each of the 5 polyhedra in turn; rotate all of the edges by an angle ϕ; pick one edge as my reference; and then compute and plot the distance of closest approach for each of the other edges as a function of rotation angle ϕ, I could apply this to any uniform-edge polyhedron, but since FIT came from the dodecahedron, I started with it to see if there were any other woven polyhedra lurking inside.

The distance plot for the dodecahedron (30 edges) is shown in figure 9.

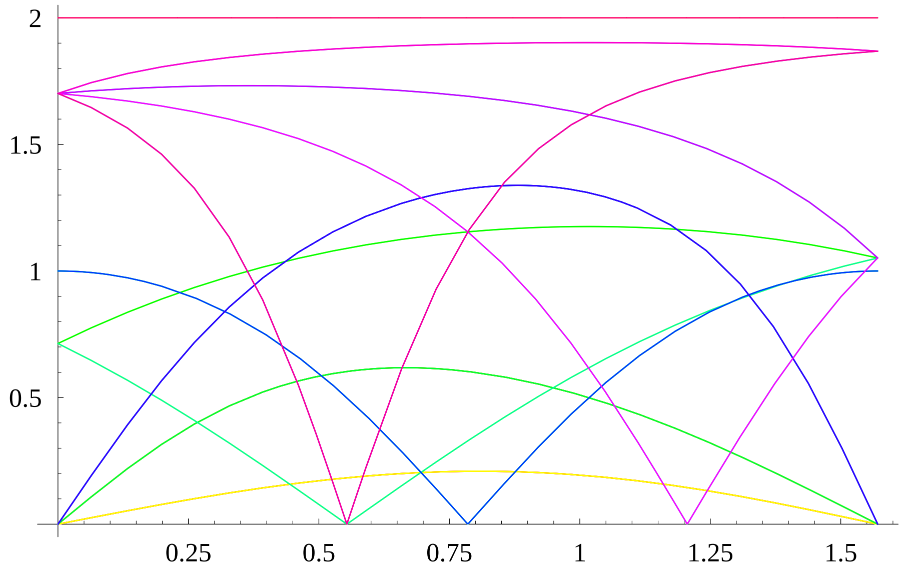


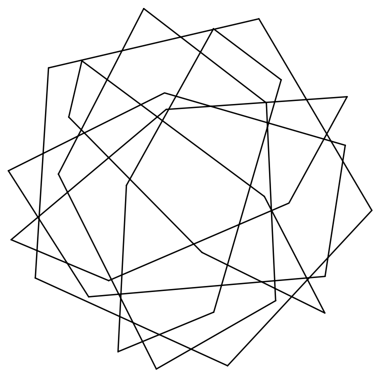
Figure 9. Distance of closest approach to one edge for all other edges, as a function of rotation angle (in radians).

Now, although this plot looks very busy, a few important points stand out. The first is that although there are 29 other edges in a dodecahedron, there are obviously fewer than 29 lines visible. That's simply because some of the lines overlap each other perfectly, i.e., there are two or more edges that maintain the same distance from the reference edge for all angles.

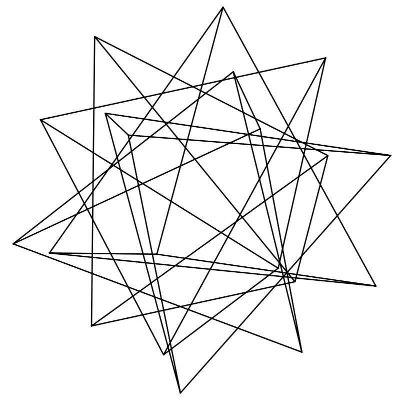
The second important point is that there are obviously angles where the distance of closest approach for some edge (or edges) goes to zero; in other words, the edges intersect. This intersection occurs at a rotation angle of 0 radians, which is (of course) the original dodecahedron; it also occurs at a rotation angle of $\pi/2$ (1.5708 radians, or 90°), which is the icosahedron; but there are also three other zero crossings, which correspond to intermediate cases. These occur at rotation angles of .553574 (31.72°), .785398 (45°), and 1.20593 (69.09°).

Each of these three zero crossings was a potential candidate for a woven polyhedron. What did they look like? I plotted the three in skeletal form.

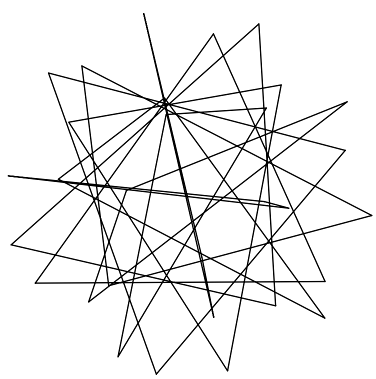
$\phi \to 0.553574$



$\phi \to 0.785398$



$\phi \to 1.20593$



The middle polyhedron should look familiar: it's our old friend, FIT. But the other two are new. Rotation angle ϕ=.55357 is a composition of 6 pentagons; rotation angle ϕ=.120593 is a composition of 10 triangles.

This was, briefly, disturbing. After all, I was looking for woven *polyhedra*. What I had found was woven *polygons*. But upon reflection, this wasn't so bad. After all, a polygon is, in a way, also a polyhedron — it's just a polyhedron with a single face. You can certainly embed a polygon in 3-dimensional space, as I have above. And if I made a polygon out of 3-dimensional sticks to form an origami burr, the polygon itself would become three-dimensional simply because it would be built from 3-dimensional solids.

So, any of the three skeletal structures shown above could serve as the basis for a woven polyhedron. The middle structure, ϕ=.785398, is the basis of Hull's FIT. But I could use the first, ϕ=.55357 as the basis for Lang's Six Intersecting Pentagons, and I could use the third, ϕ=1.20593, for Ten Intersecting Triangles.

It also became apparent that with the proliferation of such models, 3-letter abbreviations would likely lead to redundancies (does T stand for Tetrahedra or Triangle?), so I adopted the notation to denote a structure by three numbers: the number of individual polyhedra that were woven together, the number of faces in each individual polyhedron, and the number of edges in each face. Thus, Hull's FIT would be denoted 5×4×3 (5 tetrahedra, each with 4 faces, each face with 3 edges). The other two structures would therefore be 6×1×5 and 10×1×3, respectively.

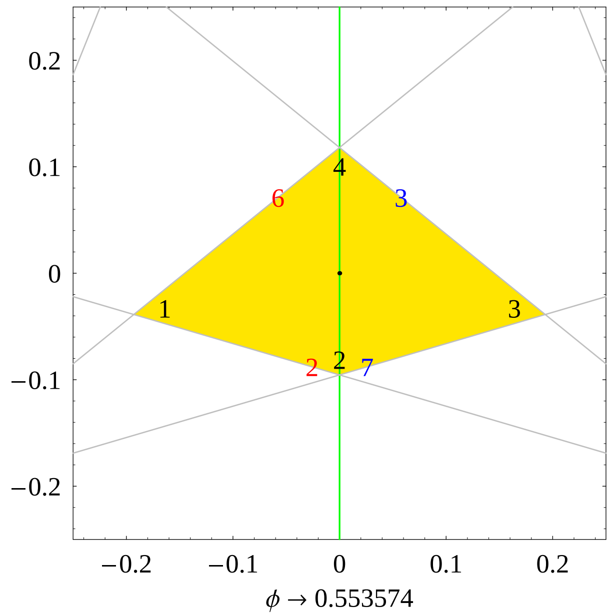
But a line drawing does not an origami model make. I now had the skeleton; I still needed to come up with the origami stick patterns themselves. That took me to the next step in the design process.

To make a rigid structure, I needed each stick to be held in place by its surrounding sticks; I needed it to touch its neighbors. The shape of the stick would be set (or at least constrained) by this requirement.

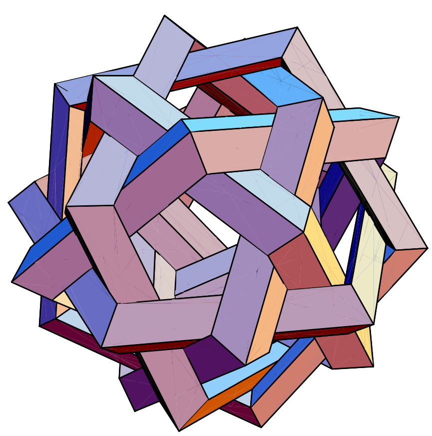
I also chose, as a design decision, that the sticks would have a constant cross section all along their length to simplify their folding. These two constraints, it turns out, set stringent limits on the shape of each stick. Imagine, for a moment, that each stick starts out infinitesimally thin, like a long, skinny balloon. We slowly inflate the balloon, inflating each stick at the same rate. As the sticks inflate, each point expands away from the center of the stick at the same rate, so the cross section of the stick is a circle. However, when two sticks touch, the balloons can't continue to expand at the point of intersection; and as the balloons continue to inflate, the point where two balloons touch expands to form a flat plane, like the intersection of two soap bubbles. At that point, the circular cross section of the stick must develop a flat spot where it touches another stick. If we keep inflating the sticks, the circular parts keep expanding away from the axis of the stick, but the flat parts just get wider and wider.

As the expansion continues, the stick will eventually bump into another stick and its cross section will develop another flat spot. As the inflation continues, the flat spots increase, multiply, and eventually merge, until a point is reached when there are no circular parts left. The stick can inflate no more; its cross section consists entirely of straight lines so that it has a polygonal cross section; and this polygon defines the largest cross section that any stick can have.

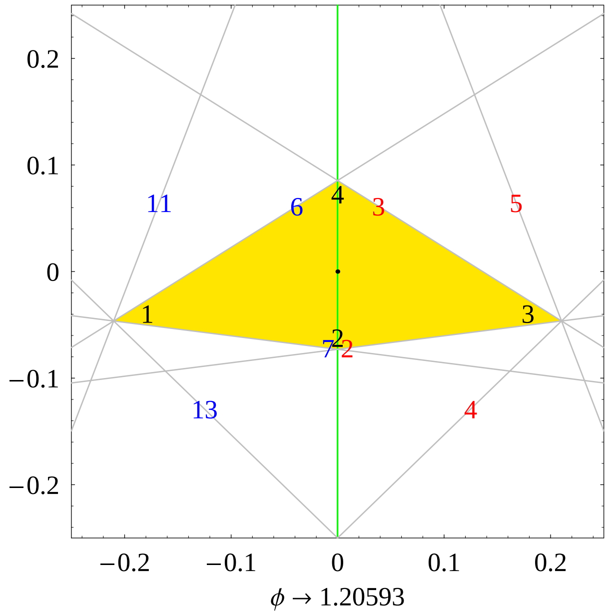
It's actually possible to carry out this thought experiment mathematically. I established a local coordinate system for each stick, then calculated the distance and direction from one stick (my "reference" stick) to each of the other sticks. If two sticks touch, the "flat place" along the cross section of the stick lies in a plane that is the perpendicular bisector of the line joining the two sticks at their point of closest approach. By plotting all such lines in a plane, I could see and construct the maximum cross section of the stick.



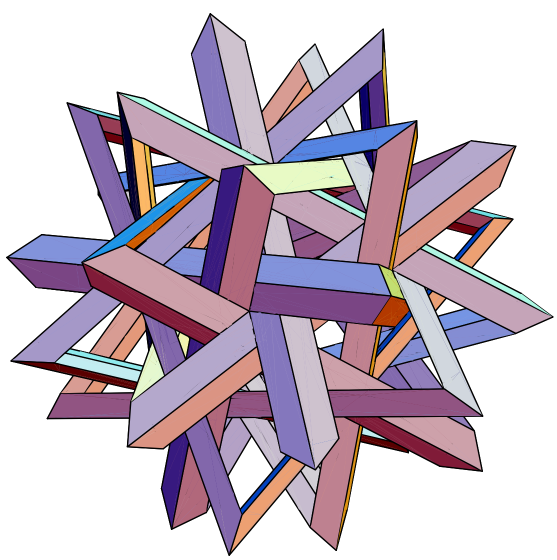
The figure above shows such a plot for the 6×1×5 structure, i.e., the ϕ=.553574 rotation angle. In this plot, you are looking down the axis of the stick corresponding to edge number 1 (out of 30 total edges). The axis of the stick is the dot in the center. ("Down" points toward the center of the polyhedron.) Each of the gray lines is the projection of the plane between edge 1 and some other edge; the red or blue number on each line is the index of the corresponding edge. The green line through the middle of the figure is the projection of the plane containing adjacent edges. The maximum size of the cross section is the yellow region, which is bounded by edges 6,2,7, and 3. If you made a polygon out of sticks with this cross section, each stick would touch 4 others along its length (in addition to being joined to two others at its ends). And sure enough, if you construct the polyhedron from sticks with this cross section, that is exactly what you get.



This is indubitably a woven polyhedron. We can do the same thing for the 10×1×3 structure. Its cross section is also a kite shape and it touches edges 6,7,2, and 3, but in a different order as you go around the perimeter of the maximum cross section.



And indeed, this, too, makes an interesting woven polyhedron.



For my money, of the two new structures, 10×1×3 is the more interesting. It has sharper points and a more lacy structure than the 6×1×5. And it even has more points (30) than the 5×4×3 (20). This structure had definite possibilities.

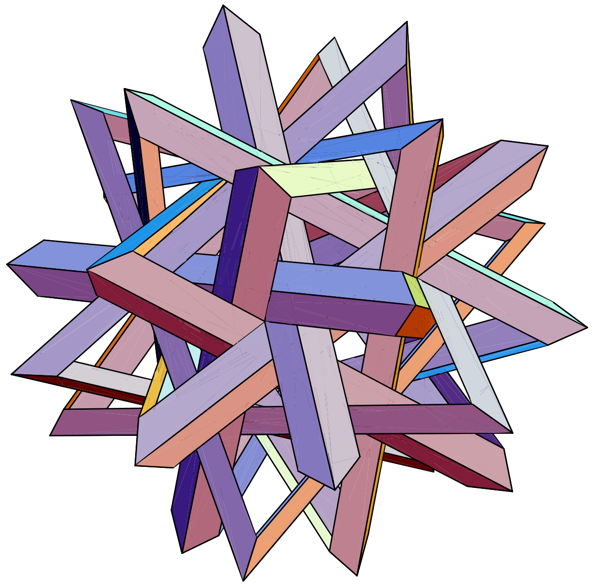
But as interesting as it is, the figure above is still just a geometric solid, a figment of my (and *Mathematica*'s) imaginations. How to render it in origami?

The key would be to figure out how to fold each stick from origami, and, hopefully, to fold it in a fairly simple fashion. I started by assuming that the stick with its kite-shaped cross section was formed from a single piece of paper; I then used *Mathematica* to figuratively unfold that sheet of paper and flatten it out.

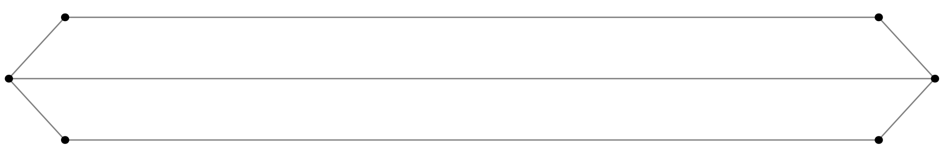
0.072949	0.244765	0.927051	0.244765
0.0607086	0.188389	0.939291	0.188389
0.	0.122383	1.	0.122383
0.0607086	0.0563762	0.939291	0.0563762
0.072949	0	0.927051	0

The figure shows an unfolded plan view of the stick. The numbers below are the x and y coordinates of the vertices at each end of the pattern (the dots in the figure above). The stick is normalized to unit length at its longest point.

This didn't present any obvious ways to start, so then I looked at whether it was possible to simplify the pattern. Indeed there was. Two of four sides of each stick face toward the interior of the model and are nearly hidden. I could eliminate them without substantially changing the appearance of the model, as shown below.

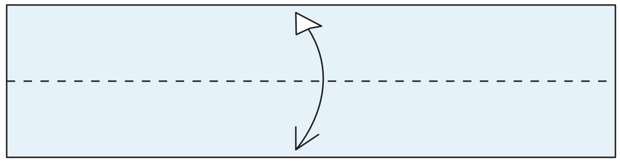


Though the appearance of the model was nearly unchanged, the pattern for the stick suddenly became very simple indeed.

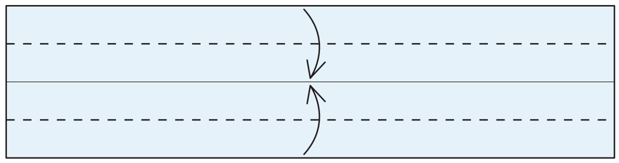


0.0607086	0.132013	0.939291	0.132013
0.	0.0660064	1.	0.0660064
0.0607086	0	0.939291	0

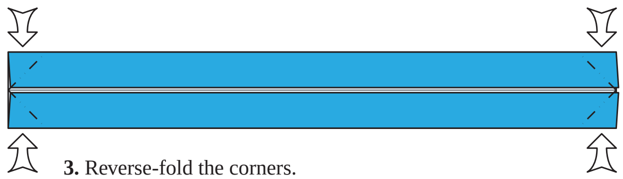
And then, two fortuitous coincidences became apparent. First, the angle at either end was very nearly a right angle, which would make it easy to construct. Second, the length-to-width ratio was very close to 15:2, which would be an easy proportion to construct. And a simple edge unit immediately suggested itself: like Ow's edge unit, take a long rectangle, fold its edges in to the center line, then reverse-fold the corners to fit the unfolded plan.



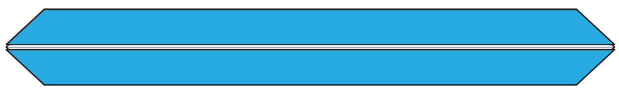
1. Begin with a 15:2 rectangle. Fold in half from top to bottom and unfold.



2. Fold the top and bottom edges to the center line.

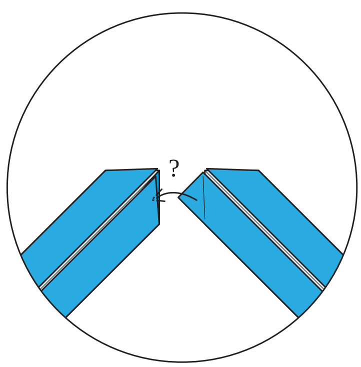


3. Reverse-fold the corners.



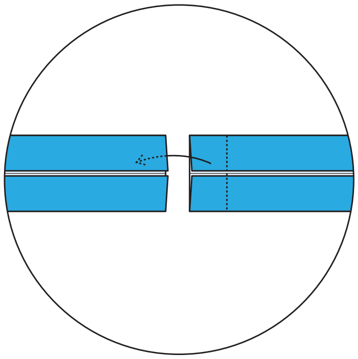
4. Finished unit.

But then a problem reared up again. How to join the units? In Ow's unit, a corner of one unit tucks into a pocket of the next unit over. However, this only works for corner angles more acute than 45° — and this unit was right at 45°. Attempting to adapt Ow's technique didn't work because the smaller corners wouldn't stay in the pockets.

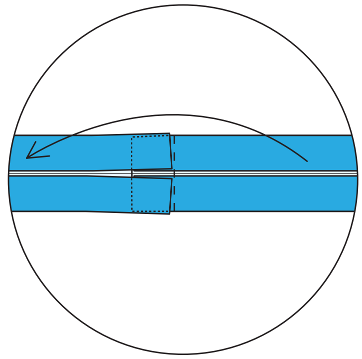


5. How to connect the units?

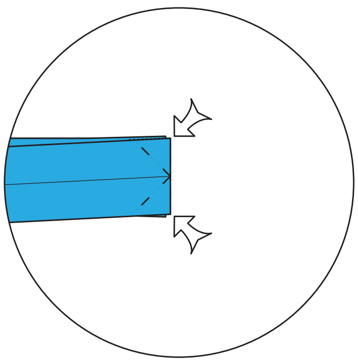
But then I realized I could actually exploit one of the differences between this structure and FIT: the fact that each of the intersecting polyhedra was just a polygonal ring. Only two edges met at a point (rather than 3 in FIT). This permitted a different method of joining two edges. In fact, if I added a bit of excess length to one end of one stick, I could slip the excess into the pocket of another and then closed-sink the corners of the combined structure to the appropriate angle. This gave an extraordinarily sturdy joint between the two ends, as shown below.



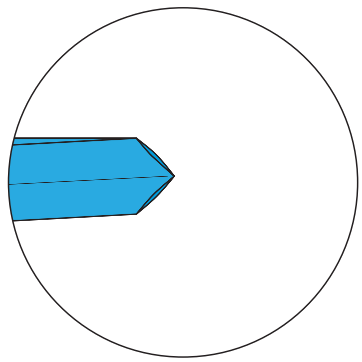
3. Instead of reverse-folding, tuck the ends of one unit inside the other.



4. Valley-fold the inner unit to the left so it lies on top of the other.

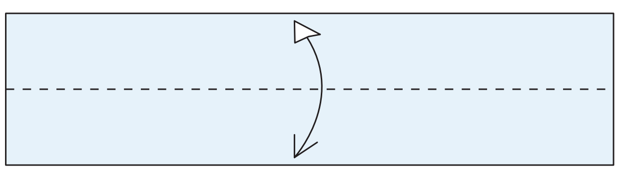


5. Closed-sink the corners. This is very hard!

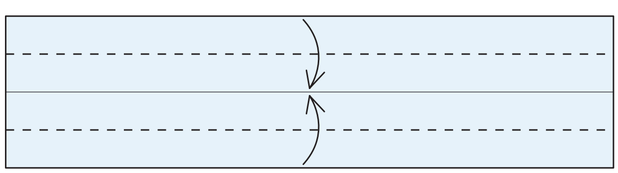


6. The units are now securely locked together.

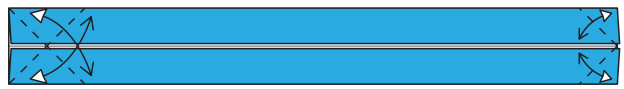
This method of joining was extraordinarily sturdy; it was also extraordinarily difficult to carry out. But whenever you have access to the inside layers of a closed sink, it is often possible to find an alternate way of forming the sink. Such was the case here, and I was able to replace the closed sink with an easier sequence of folds. I also got lucky again on the proportions: the extra bit added to tuck into the pocket transformed the length:width ratio from 15:2 to 16:2, or 4:1, which is easily obtained by dividing a square into fourths. The folding sequence for the newly modified edge unit, from a 4:1 rectangle, is shown below.



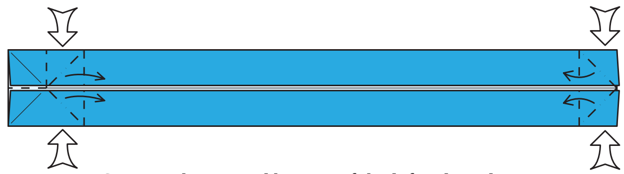
1. Begin with a 4:1 rectangle. Fold in half from top to bottom and unfold.



2. Fold the top and bottom edges to the center line.



3. Fold and unfold the diagonals on the left. Fold and unfold two corners on the right.



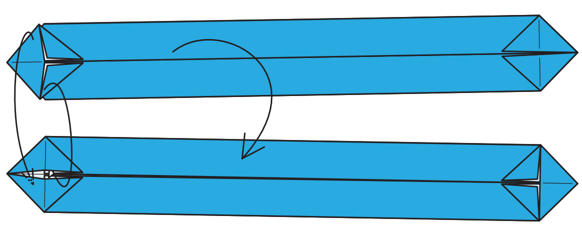
4. Squeeze the top and bottom of the left side and swing the excess paper upward. On the right, squash-fold top and bottom corners.

5. Squash-fold the left side.

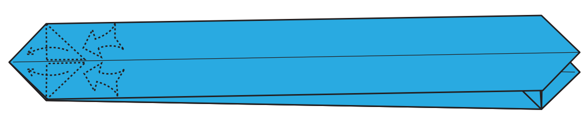
6. Valley-fold two corners.

7. Finished unit.

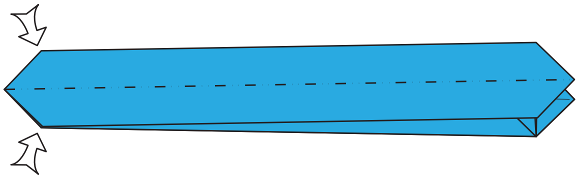
Joining the edge units is still a bit of a trick. There are two types of ends. You tuck the flaps from one type into the pockets of the other type, then reverse-fold two hidden corners into the interior of the model. After doing this, pinch the sides, and the edge strut pops into three-dimensionality.



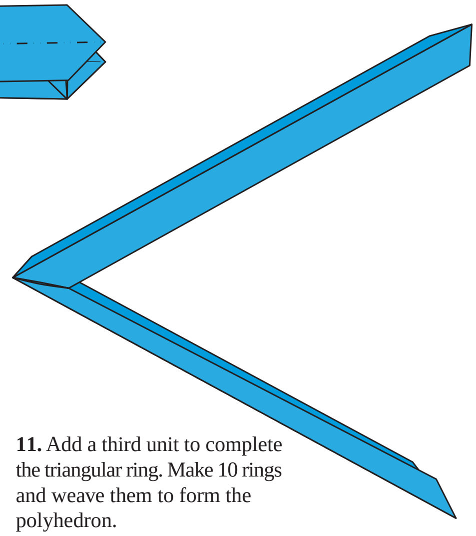
8. To join two units at the left end, tuck the flaps of one unit into the pockets of the other unit.



9. This part is still hard; reverse-fold the two hidden corners one at a time into the interior of the model.



10. Squeeze the sides of the flap to make it three-dimensional.

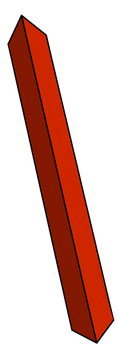


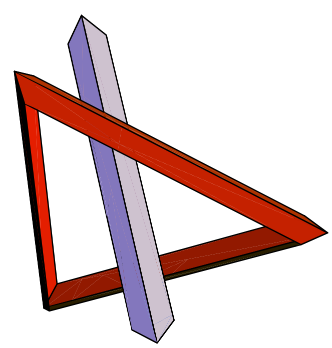
11. Add a third unit to complete the triangular ring. Make 10 rings and weave them to form the polyhedron.

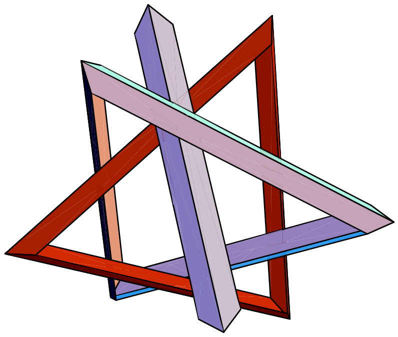
Putting three such units together gives a triangular ring.

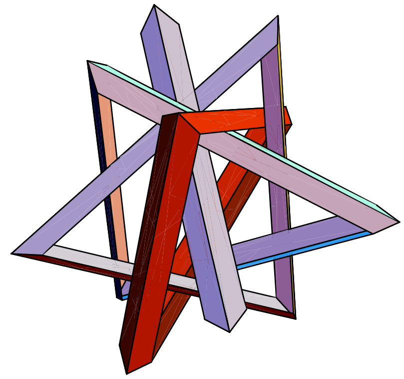
And suddenly, before I knew it, the recipe for my modular was completed. Simply take 30 1×4 strips of paper, make a few simple folds in each; lock them into 10 triangular rings; then undo and weave the triangles together to make the woven polyhedron.

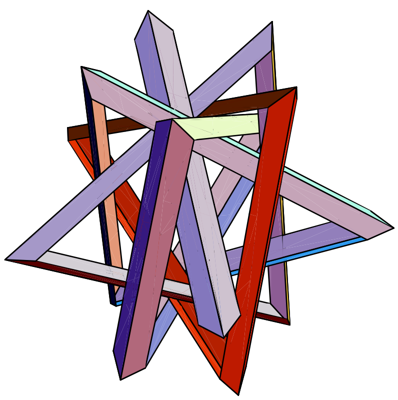
That is, however, easier said than done. The problem with assembly is that each ring must be precisely looped around all the others. As you add more and more rings to the structure, it gets harder and harder to keep all the rings in their proper spot. The next 11 figures show the sequential assembly of the structure.

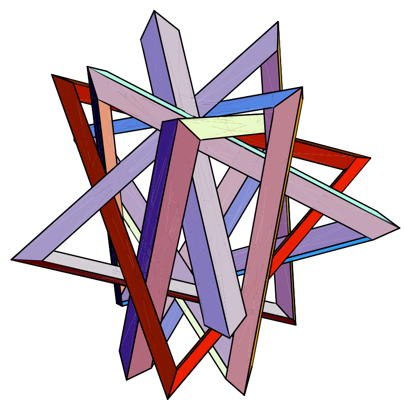


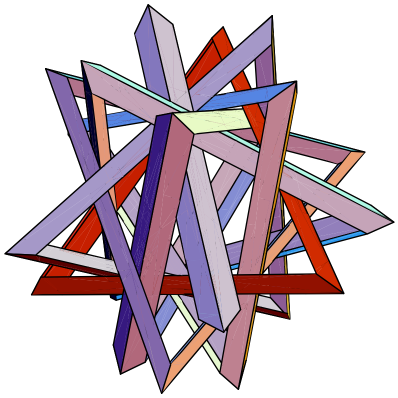


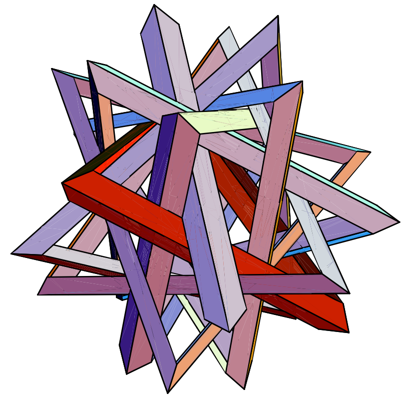


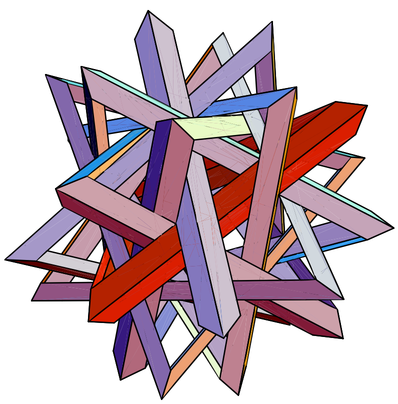


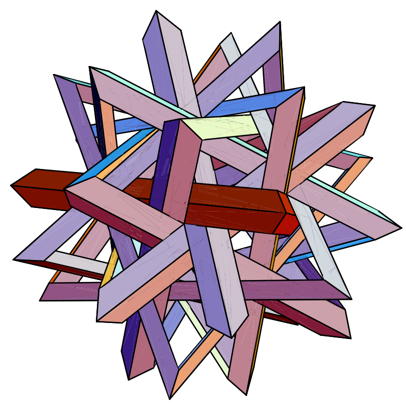


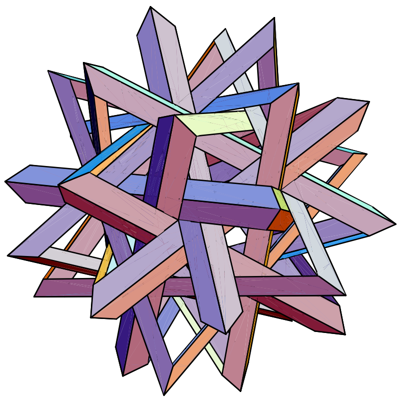












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And so, it appeared that with this model, I had finally reached my goal of coming up with something new. Both the 10×1×3 structure and my edge unit were (to my knowledge, at least) original. Furthermore, the assembly of the modular presents some fairly stiff challenges, and while I don't actually seek to make my designs difficult to fold, I must admit to a small thrill of excitement when they turn out that way.

It was a pleasant coincidence that the proportions of the strip came out to be an integral ratio, 1:4. A small integral proportion is a property shared with Hull's FIT, for which the mathematically exact proportion is 1:3.0230, but 1:3 is close enough for all practical purposes. The mathematical ideal for my unit was slightly more that 1:4.04, but 1:4 is good enough for all practical purposes and is easy to construct to boot.

In this version, each triangle is made from three 1:4 rectangles. But since the edge units are joined in a ring, you could also use a single long strip of paper whose ends get joined using the same locking mechanism. If you go this route, then you would need 10 strips, each in the proportions 1:23, to make the complete polyhedron. How to fold the strips — and how to form the two corners other than the one where the two ends lock — are left as an exercise for the reader.

Next issue: more woven polyhedra!