

# Polypolyhedra in Origami

Robert J. Lang  
Rjlang@home.com

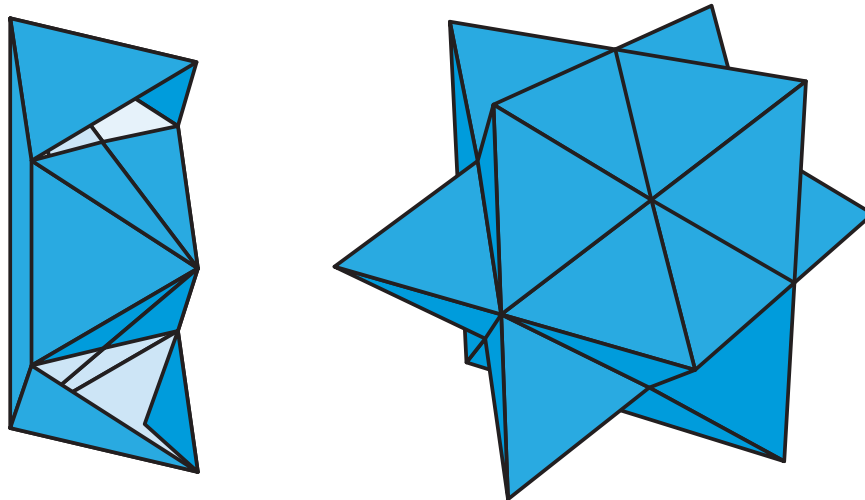
3rd International Conference on Origami in  
Science, Math, & Education

# Background

- Origami modulars: many identical pieces that hold themselves together, usually forming a geometric solid
- A “burr puzzle” is a wooden puzzle made of sticks that lock together, usually forming a geometric solid
- Can one make an origami burr puzzle?
- Yes...

# Origami Burrs

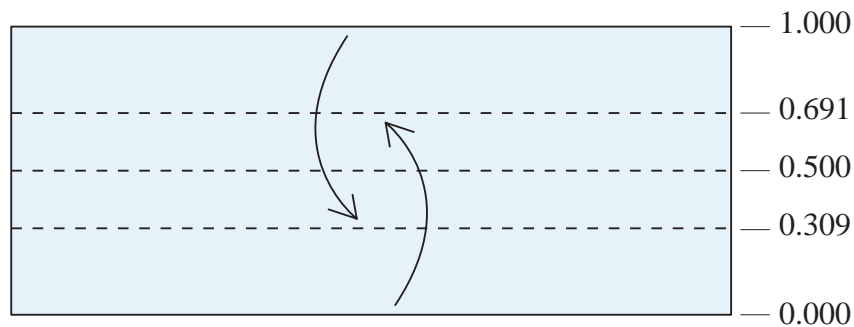
- Peter Ford made several back in the '80s
- Here is a rendition of the “classic six-piece burr”



To be published in “Puzzlers Tribute,” David Wolfe, ed. (2001: A. K. Peters)

# More Burrs

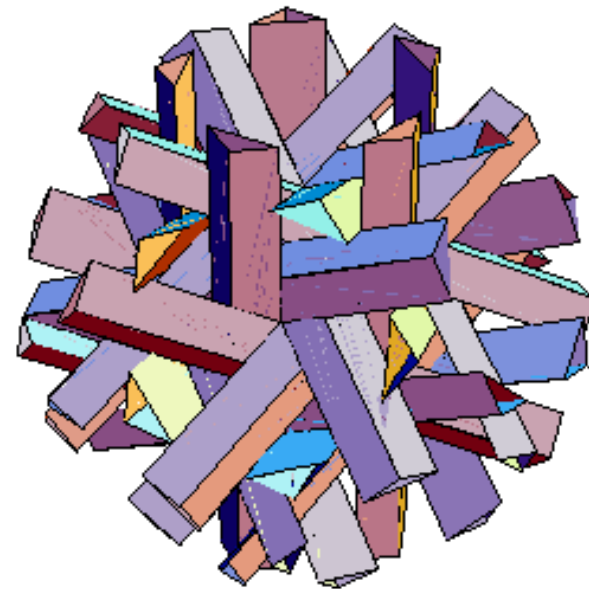
- Can one make a burr with more (but identical) pieces?
- Yes...(but there's one problem)...



1. Begin with a dollar bill. Mark off three horizontal creases at the divisions shown and fold on all three creases so that the ends overlap one another.



2. Finished unit. The stick has a triangular cross section.

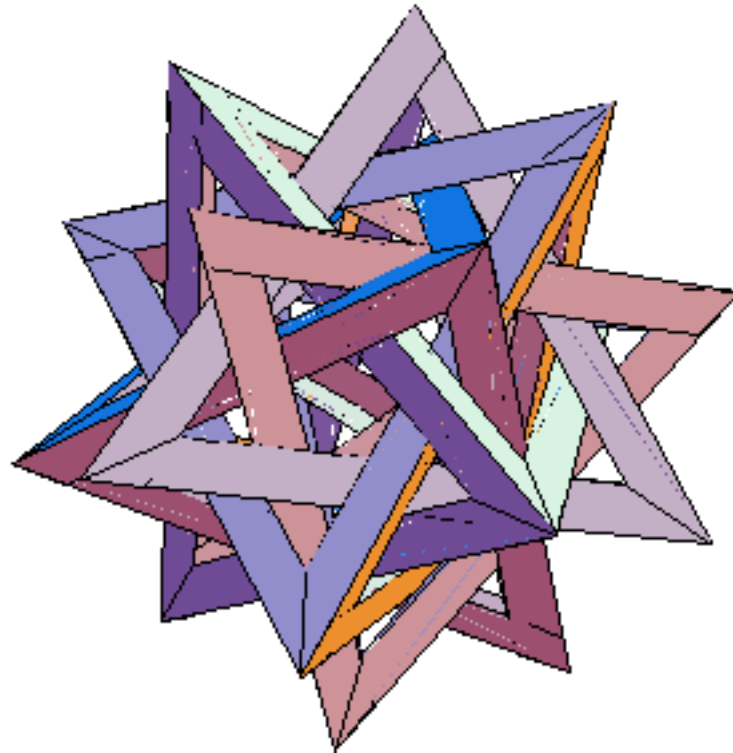


# More Burrs

- It collapses!
- No burr composed of convex polyhedra is stable against all possible rigid-body motions.
- What if we joined the sticks at their ends?
- Individual groups of sticks would form polyhedra. The burr would be a collection of interlocking polyhedra.
- Only one problem...

# More Burrs

- It's been done!
- Tom Hull's "Five Intersecting Tetrahedra," using Francis Ow's edge unit.



# Polypolyhedra

- Are there others?
- What are we looking for?
- A Polypolyhedron is:
  - a compound of multiple linked polyhedral skeletons with uniform nonintersecting edges
  - “Uniform” = all alike
  - The edges are 1-uniform (1 type of edge)
  - The vertices could be 1-uniform or 2-uniform. If 2-uniform, every edge has 1 of each kind of vertex
- How many topologically distinct polypolyhedra are there?

# Polypolyhedra

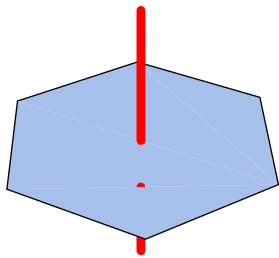
- Prior art
- Five intersecting tetrahedra is a well-known geometrical construction
- “Orderly Tangles” by A. Holden identifies numerous compounds of polygons and polyhedra, which he calls “polylinks”
- No complete listing known

# Polypolyhedra & Symmetry

- We start by classifying polypolyhedra according to their symmetry
- If all edges are “alike”, then any one edge can be transformed into any other edge by some solid rotation
- Any polypolyhedron can be labeled by the set of rotations that maps any one edge into all others
- The set of rotations for any given polypolyhedron form a Group.

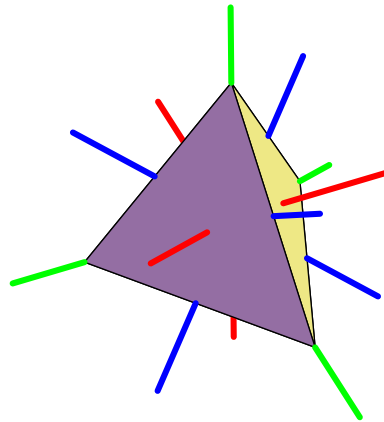
# Rotation Groups

- There are 5 families of rotation groups in 3-D



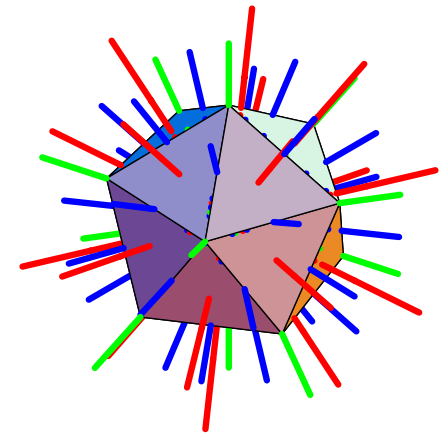
Cyclic group

Order  $n$



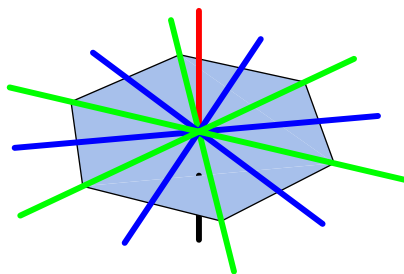
Tetrahedral group

Order 12



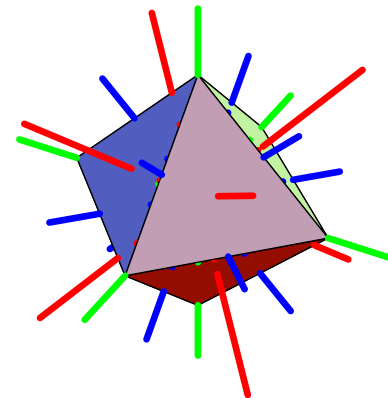
Icosahedral group

Order 60



Dihedral group

Order  $2n$

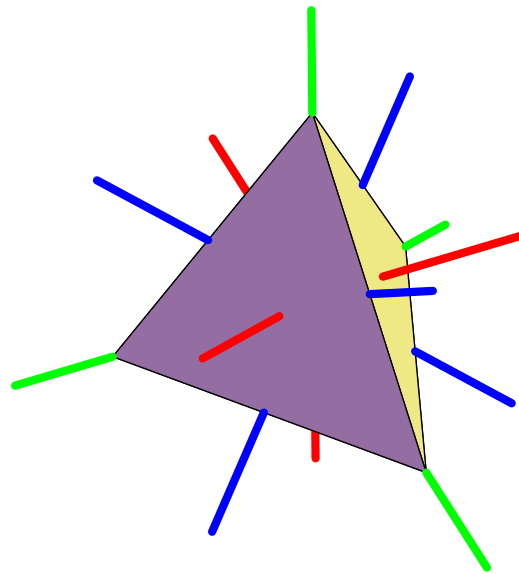


Octahedral group  
Order 24

ence on Origami in  
: Education

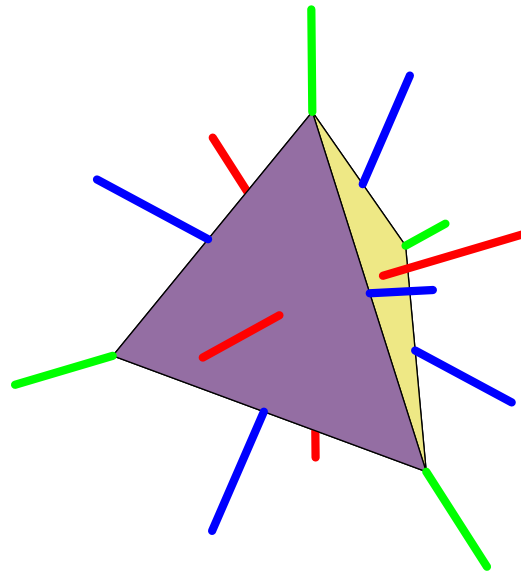
# Orbits

- The set of points obtained by applying all rotation operators to a single point is called an “orbit”
- The number of distinct points in an orbit is called the order of the orbit
- The maximum order of an orbit is the order of the rotation group.



# Orbits

- If a point is unchanged by a rotation, it lies on an axis of rotation
- The orbits of points on axes of rotation have lower order than the group order
- The tetrahedron has orbits of order 4 (red), 4 (green), 6 (blue), and 12 (everything else).
- Label an orbit by its type: Vertex, Edge, Face, or Complete (V,E,F,C)



# Orbits

- Every vertex of every edge of a polypolyhedron must lie in an orbit
- We can classify polypolyhedra by their rotational symmetry and by the orbits of the two vertices of any edge.

	V	E	F	C
Cyclic (n)			1	n
Dihedral (n)	n	n	1	2 n
Tetrahedral	4	6	4	12
Octahedral	6	12	8	24
Icosahedral	12	30	20	60

# Orbital Types

- Label the edge by a symbol  $\{o_1, o_2\}$  of the orbital types of its vertices
- For example,  $\{V, C\}$  is an edge with one vertex in a  $V$  (vertex) orbit, the other in a  $C$  (complete) orbit
- Append a number if the orbits are of the same type but different orbits:  $\{V_1, V_1\}$  mean the vertices are in two different  $V$  orbits

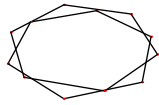
# Orbital Types

- Enumerate orbits
- Eliminate redundancies, e.g.,  $\{V,C\}$  is the same as  $\{C,V\}$
- Eliminate orbits that lead to 1-valent edges (disjoint sticks), e.g.,  $\{V,C\}$
- There are now only 10 distinct types of edges that lead to interesting polypolyhedra:  $\{V1,V1\}$ ,  $\{V1,V2\}$ ,  $\{V1,E1\}$ ,  $\{V1,F1\}$ ,  $\{E1,E1\}$ ,  $\{E1,E2\}$ ,  $\{E1,F1\}$ ,  $\{F1,F1\}$ ,  $\{F1,F2\}$ , and  $\{C,C\}$ .

# Seed Edges

- For each orbital type  $\{o_1, o_2\}$ , pick a single vertex in  $o_1$ .
- Every possible choice of  $o_2$  vertex gives a different “seed edge.”
- Apply all rotation operators to get all other edges.
- Eliminate duplicates
- There are 41 distinct families!

DCC 6, 2-2, 2x1x6



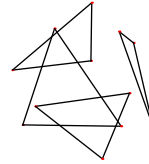
TEI E2 3, 2-2, 3x1x4



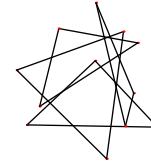
TCC 7, 2-2, 4x1x3



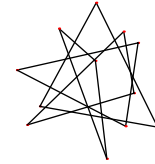
TCC 2, 2-2, 4x1x3



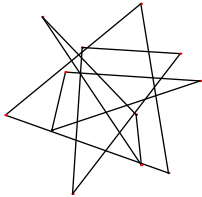
TCC 10, 2-2, 4x1x3



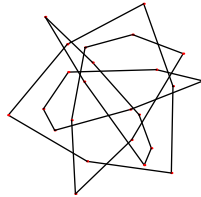
TCC 12, 2-2, 4x1x3



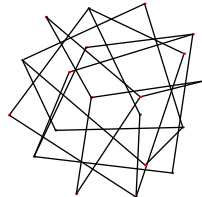
O EI E1 5, 2-2, 4x1x3



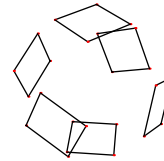
O EI E2 2, 2-2, 4x1x6



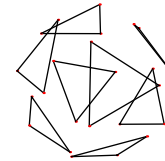
O EI F1 4, 2-3, 4x3x4



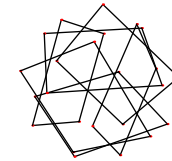
OCC 4, 2-2, 6x1x4



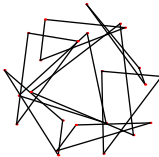
OCC 2, 2-2, 8x1x3



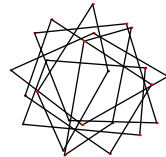
OCC 11, 2-2, 6x1x4



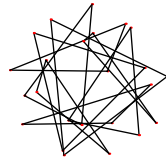
OCC 9, 2-2, 8x1x3



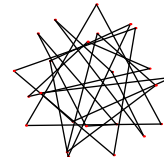
OCC 6, 2-2, 6x1x4



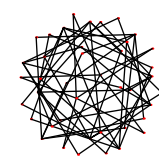
OCC 8, 2-2, 8x1x3



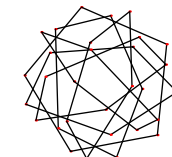
OCC 14, 2-2, 8x1x3



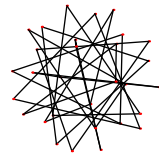
IV I E1 4, 5-2, 6x5x4



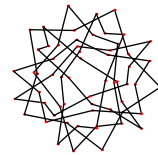
IEI E1 5, 2-2, 6x1x5



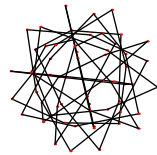
IEI E1 24, 2-2, 10x1x3



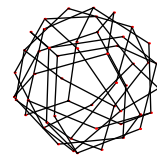
IEI E2 6, 2-2, 6x1x10



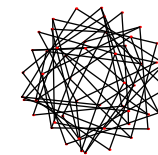
IEI E2 20, 2-2, 10x1x6



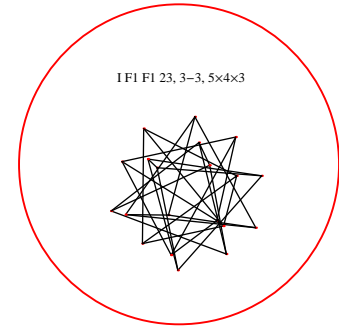
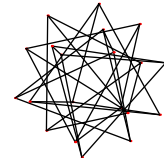
IEI F1 6, 2-3, 5x4x6



IEI F1 7, 2-3, 10x3x4



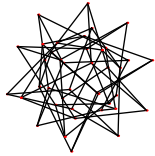
IEI F1 23, 3-3, 5x4x3



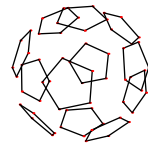
Recognize this?

3rd International Conference on Origami in  
Science, Math, & Education

IF1 F2 7, 3-3, 5x6x4



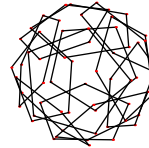
ICC 14, 2-2, 12x1x5



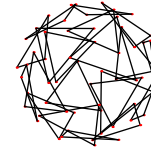
ICC 3, 2-2, 20x1x3



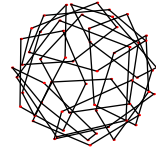
ICC 6, 2-2, 12x1x5



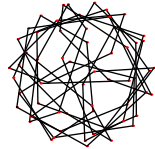
ICC 32, 2-2, 20x1x3



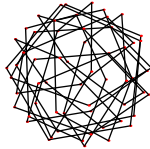
ICC 4, 2-2, 12x1x5



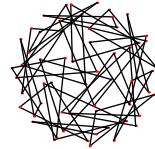
ICC 28, 2-2, 12x1x5



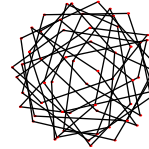
ICC 9, 2-2, 12x1x5



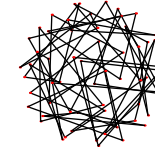
ICC 12, 2-2, 20x1x3



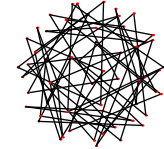
ICC 11, 2-2, 12x1x5



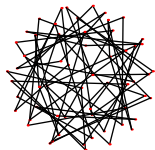
ICC 40, 2-2, 20x1x3



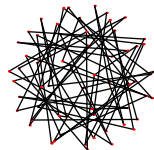
ICC 60, 2-2, 20x1x3



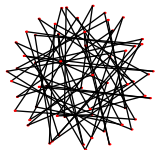
ICC 8, 2-2, 20x1x3



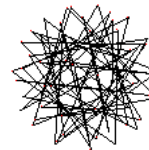
ICC 35, 2-2, 20x1x3



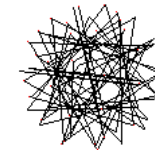
ICC 24, 2-2, 20x1x3



ICC 23, 2-2, 20x1x3



ICC 36, 2-2, 20x1x3

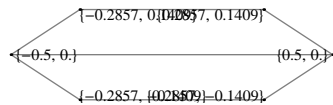
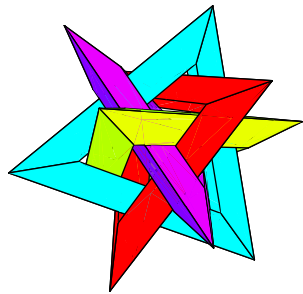


# From Skeleton to Origami

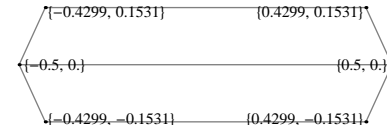
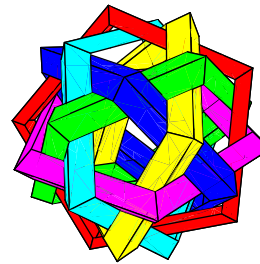
- Not all polypolyhedra are linked; eliminate unlinked
- Some belong to families characterized by 1 or 2 continuously-variable parameters
- But, we can enumerate them all!
  
- Replace each line in the skeleton with the widest possible edge unit (e.g., an Ow unit)

# Family 1: Discrete

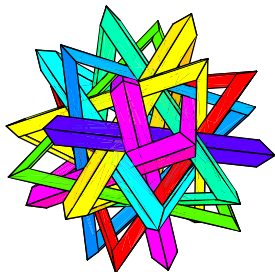
- There are 4 distinct homoorbital polypolyhedra



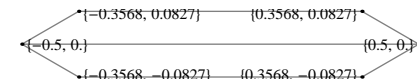
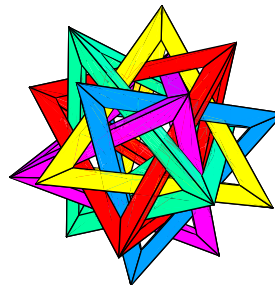
Four Intersecting Triangles (4x1x3)



Six Intersecting Pentagons (6x1x5)



Ten Intersecting Triangles (10x1x3)



Five Intersecting Tetrahedra (5x4x3)

# Family 2: 1-parameter variation

- Hetero-orbital polypolyhedra are 2-uniform, 2 types of vertex composing two orbits
- Each forms a continuously varying family characterized by the ratio of their orbital radii
- Different orbital ratios can be topologically distinct (“varieties”)
  - T {E1 E2} - (3x1x4) - 1 variety
  - O {E1 E2} - (4x1x6) - 1 variety
  - O {E1 F1} - (4x3x4) - 2 varieties
  - I {V1 E1} - (6x10x4) - 4 varieties
  - I {E1 F1} - (5x4x6) - 2 varieties
  - I {E1 F1} - (10x3x4) - 3 varieties

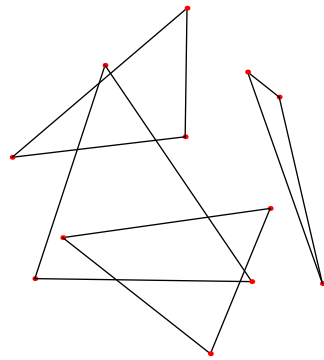
# Family 3: 2-parameter variation

- The orbital seed vertex of a  $\{C,C\}$  polypolyhedra can vary in 2 dimensions
- There are 5 families, each corresponding to a manipulation of a Platonic solid
- Shift each face outward by a factor  $\rho$ , rotate through an angle  $\phi$ .
- Plot the distance of closest approach between any two edges versus  $\rho$  and  $\phi$ .

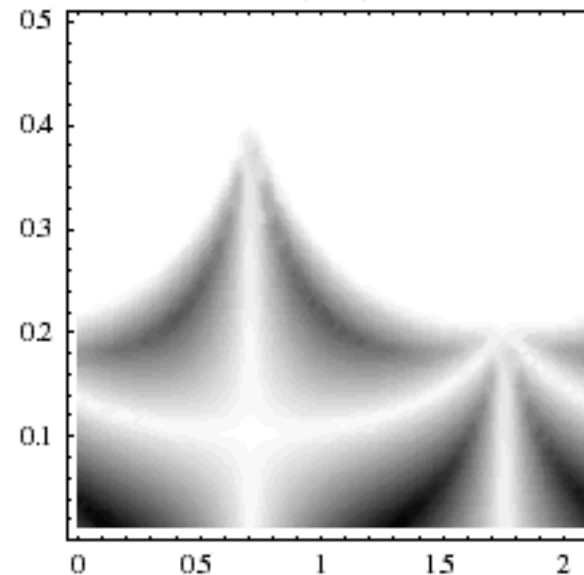
# {C,C} Tetrahedron

- Shift and rotate each face
- Plot the distance of closest approach between edges
- Each dark region is a topologically distinct polypolyhedron

TCC 2, 2-2, 4x1x3

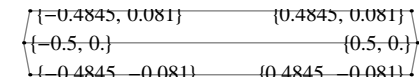
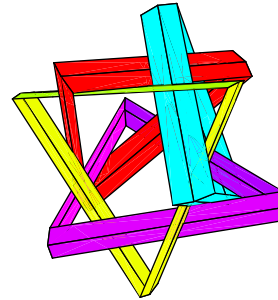
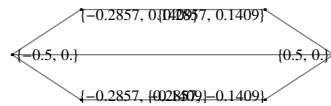
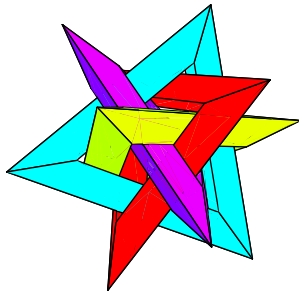


TCC 2, 2-2, 4x1x3



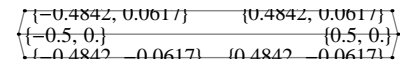
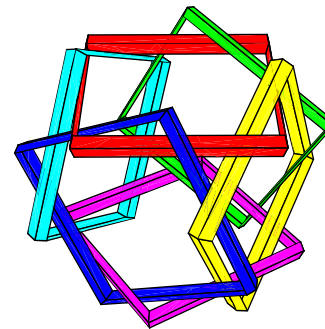
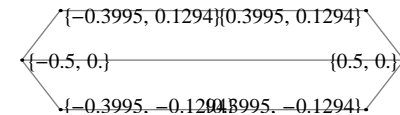
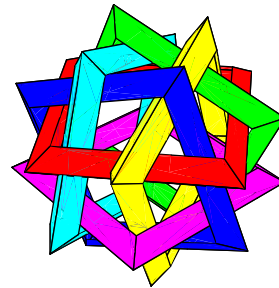
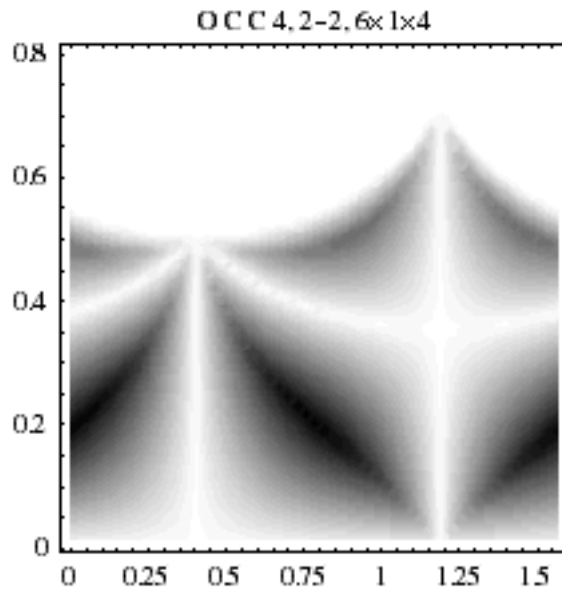
# {C,C} Polytetrahedron

- There are 2 topologically distinct {C,C} polytetrahedra.

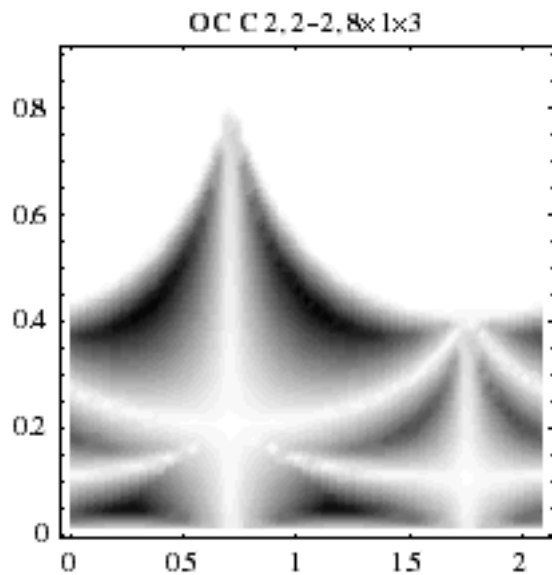


# {C,C} Polyhexahedra

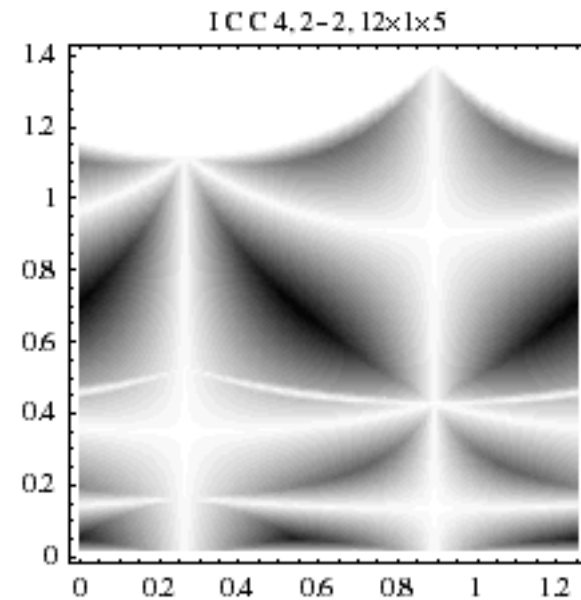
- There are 2 distinct Polyhexahedra (6x1x4).



# Polyoctahedra, Polydodecahedra



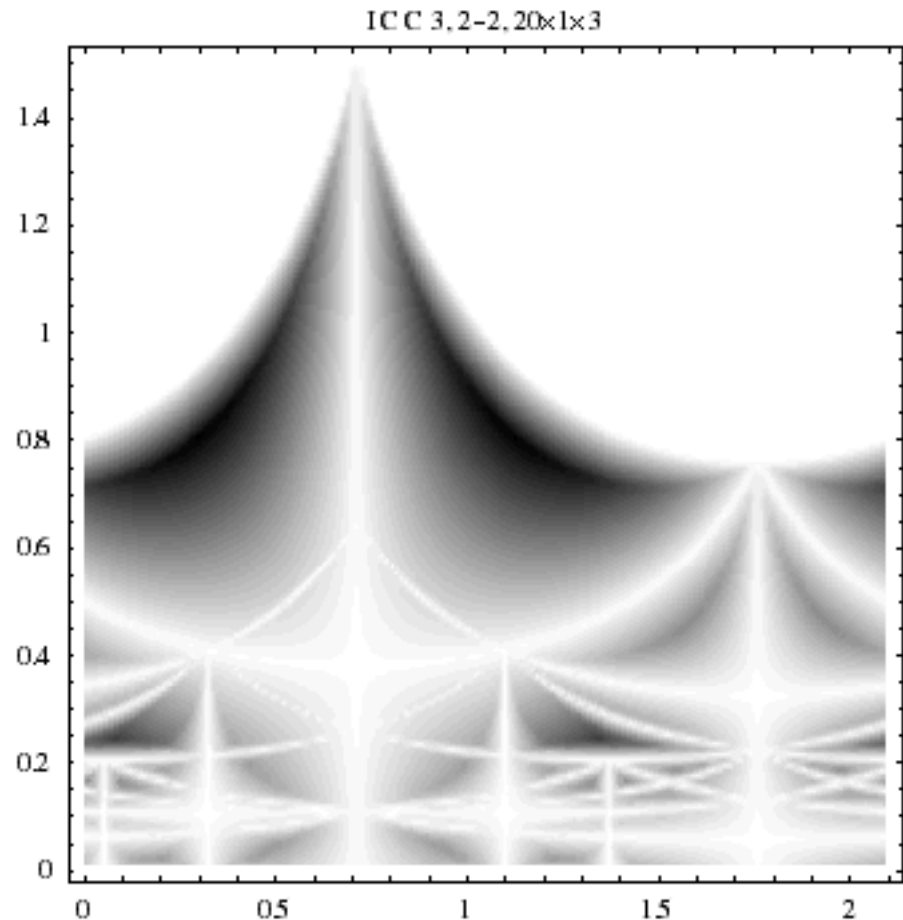
3 varieties of polyoctahedra



5 varieties of polydodecahedra

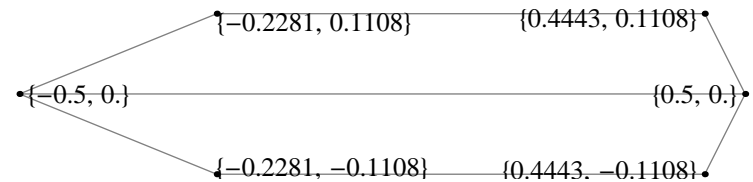
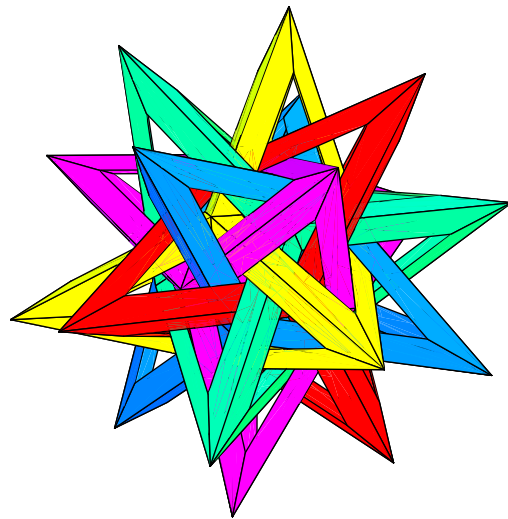
# Polyicosahedra

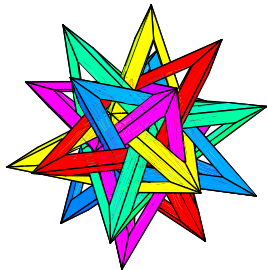
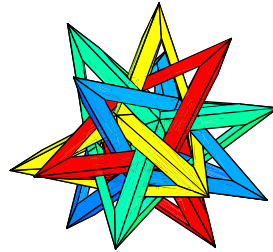
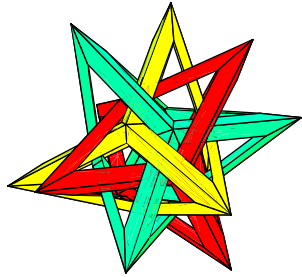
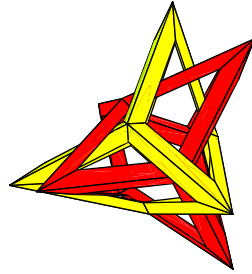
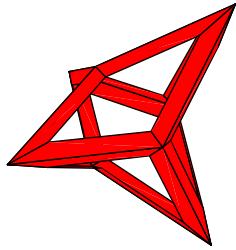
- There are 23 polyicosahedra
- A total of 54 polypolyhedra!
- Nearly all have 2-valent vertices (polygons, rather than polyhedra) with two exceptions:
  - Five intersecting tetrahedra
  - And...



# Five Intersecting Rhombic Hexahedra

- I {F1 F2}, 5x6x4, sixty identical edge units





Instructions: OrigamiUSA 2000 Annual  
Convention Program

First Full Assembly: Peter Budai (photo)

3rd International Conference on Origami in  
Science, Math, & Education