Polypolyhedra in Origami

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Background

- Origami modulars: many identical pieces that hold themselves together, usually forming a geometric solid
- A “burr puzzle” is a wooden puzzle made of sticks that lock together, usually forming a geometric solid
- Can one make an origami burr puzzle?

- Yes...
Origami Burrs

- Peter Ford made several back in the ‘80s
- Here is a rendition of the “classic six-piece burr”

More Burrs

• Can one make a burr with more (but identical) pieces?
• Yes…(but there’s one problem)...

1. Begin with a dollar bill. Mark off three horizontal creases at the divisions shown and fold on all three creases so that the ends overlap one another.

2. Finished unit. The stick has a triangular cross section.
More Burrs

• It collapses!

• No burr composed of convex polyhedra is stable against all possible rigid-body motions.

• What if we joined the sticks at their ends?

• Individual groups of sticks would form polyhedra. The burr would be a collection of interlocking polyhedra.

• Only one problem…
More Burrs

- It’s been done!
- Tom Hull’s “Five Intersecting Tetrahedra,” using Francis Ow’s edge unit.
Polypolyhedra

• Are there others?
• What are we looking for?
• A Polypolyhedron is:
  – a compound of multiple linked polyhedral skeletons with uniform nonintersecting edges
  – “Uniform” = all alike
  – The edges are 1-uniform (1 type of edge)
  – The vertices could be 1-uniform or 2-uniform. If 2-uniform, every edge has 1 of each kind of vertex

• How many topologically distinct polypolyhedra are there?
Polypolyhedra

• Prior art
• Five intersecting tetrahedra is a well-known geometrical construction
• “Orderly Tangles” by A. Holden identifies numerous compounds of polygons and polyhedra, which he calls “polylinks”
• No complete listing known
Polypolyhedra & Symmetry

- We start by classifying polypolyhedra according to their symmetry
- If all edges are “alike”, then any one edge can be transformed into any other edge by some solid rotation
- Any polypolyhedron can be labeled by the set of rotations that maps any one edge into all others
- The set of rotations for any given polypolyhedron form a Group.
Rotation Groups

- There are 5 families of rotation groups in 3-D

- Cyclic group
  Order $n$

- Dihedral group
  Order $2n$

- Tetrahedral group
  Order 12

- Octahedral group
  Order 24

- Icosahedral group
  Order 60
Orbits

- The set of points obtained by applying all rotation operators to a single point is called an “orbit”
- The number of distinct points in an orbit is called the order of the orbit
- The maximum order of an orbit is the order of the rotation group.
Orbits

- If a point is unchanged by a rotation, it lies on an axis of rotation
- The orbits of points on axes of rotation have lower order than the group order
- The tetrahedron has orbits of order 4 (red), 4 (green), 6 (blue), and 12 (everything else).
- Label an orbit by its type: Vertex, Edge, Face, or Complete (V,E,F,C)
Orbits

- Every vertex of every edge of a polypolyhedron must lie in an orbit
- We can classify polypolyhedra by their rotational symmetry and by the orbits of the two vertices of any edge.

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Orbital Types

• Label the edge by a symbol \(\{o_1,o_2\}\) of the orbital types of its vertices

• For example, \(\{V,C\}\) is an edge with one vertex in a V (vertex) orbit, the other in a C (complete) orbit

• Append a number if the orbits are of the same type but different orbits: \(\{V_1,V_1\}\) mean the vertices are in two different V orbits
Orbital Types

• Enumerate orbits
• Eliminate redundancies, e.g., \{V,C\} is the same as \{C,V\}
• Eliminate orbits that lead to 1-valent edges (disjoint sticks), e.g., \{V,C\}

• There are now only 10 distinct types of edges that lead to interesting polypolyhedra: \{V1,V1\}, \{V1,V2\}, \{V1,E1\}, \{V1,F1\}, \{E1,E1\}, \{E1,E2\}, \{E1,F1\}, \{F1,F1\}, \{F1,F2\}, and \{C,C\}. 
Seed Edges

- For each orbital type \(\{o_1,o_2\}\), pick a single vertex in \(o_1\).
- Every possible choice of \(o_2\) vertex gives a different “seed edge.”
- Apply all rotation operators to get all other edges.
- Eliminate duplicates

- There are 41 distinct families!
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From Skeleton to Origami

- Not all polypolyhedra are linked; eliminate unlinked
- Some belong to families characterized by 1 or 2 continuously-variable parameters
- But, we can enumerate them all!

- Replace each line in the skeleton with the widest possible edge unit (e.g., an Ow unit)
Family 1: Discrete

- There are 4 distinct homoorbital polypolyhedra

Four Intersecting Triangles (4x1x3)

Six Intersecting Pentagons (6x1x5)

Ten Intersecting Triangles (10x1x3)

Five Intersecting Tetrahedra (5x4x3)
Family 2: 1-parameter variation

- Heteroorbital polypolyhedra are 2-uniform, 2 types of vertex composing two orbits
- Each forms a continuously varying family characterized by the ratio of their orbital radii
- Different orbital ratios can be topologically distinct ("varieties")
  - T \{E1 \ E2\} - (3x1x4) - 1 variety
  - O \{E1 \ E2\} - (4x1x6) - 1 variety
  - O \{E1 \ F1\} - (4x3x4) - 2 varieties
  - I \{V1 \ E1\} - (6x10x4) - 4 varieties
  - I \{E1 \ F1\} - (5x4x6) - 2 varieties
  - I \{E1 \ F1\} - (10x3x4) - 3 varieties
Family 3: 2-parameter variation

- The orbital seed vertex of a \{C,C\} polypolyhedra can vary in 2 dimensions
- There are 5 families, each corresponding to a manipulation of a Platonic solid
- Shift each face outward by a factor $\rho$, rotate through an angle $\phi$.
- Plot the distance of closest approach between any two edges versus $\rho$ and $\phi$. 
\{C,C\} Tetrahedron

- Shift and rotate each face
- Plot the distance of closest approach between edges
- Each dark region is a topologically distinct polypolyhedron

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{C,C} Polytetrahedron

• There are 2 topologically distinct {C,C} polytetrahedra.
{C,C} Polyhexahedra

- There are 2 distinct Polyhexahedra (6x1x4).
Polyoctahedra, Polydodecahedra

3 varieties of polyoctahedra
5 varieties of polydodecahedra
Polyicosahedra

- There are 23 polyicosahedra
- A total of 54 polypolyhedra!
- Nearly all have 2-valent vertices (polygons, rather than polyhedra) with two exceptions:
  - Five intersecting tetrahedra
  - And…
Five Intersecting Rhombic Hexahedra

- I \{F1 F2\}, 5x6x4, sixty identical edge units
Instructions: OrigamiUSA 2000 Annual Convention Program

First Full Assembly: Peter Budai (photo)